Axiomatization of Group Contest Success Functions

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Abstract

Group contest success functions are axiomatized for general production functions, focusing particularly on the additive production function and the multiplicative Cobb-Douglas production function. Essential for axioms supporting the additive production function, driven by substitutability across efforts, is the sufficiency of only one group member exerting effort. Essential for axioms supporting the multiplicative production function, driven by complementarity across efforts, is that all group members exert efforts. The additive production function is further supported by an axiom where adding an amount to one effort and subtracting the same amount from a second equivalent substitutable effort does not change the winning probabilities. The Cobb-Douglas production function is supported by a strong homogeneity axiom where an equiproportionate change in matched group member's efforts does not affect the winning probabilities.

1 Introduction

Contest axiomatization started with Skaperdas (1996), extended to asymmetric contests by Clark and Riis (1998). Subsequently Münster (2009) axiomatized group contest success functions, observing that competition for rents such licenses, privileges, monopoly opportunities, election opportunities, public goods, and R&D budgets frequently occurs between groups.[1] Further, Arbatskaya and Mialon (2010) axiomatized multi-activity contests leading to the Cobb-Douglas production function, and Hausken (2016) axiomatized multi-activity contests leading to the additive production function. This paper axiomatizes group contest success functions for general production functions accounting for addition, exponentiation, and multiplication, as expressed by the additive production function and the Cobb-Douglas production function.

A common example in the rent seeking literature, e.g. by Krueger (1974), is that a firm may seek rents through improved efficiency or lobbying. These rent seeking efforts are additive if rent seeking is operational if one of these efforts is present, regardless of whether the other effort is present or not. In contrast, the rent seeking efforts are multiplicative, e.g. as in the Cobb-Douglas production function, if rent seeking is operational if both efforts are present.

When focusing exclusively on the contest success function,[2] additive efforts can be interpreted as substitutable, so that employing more of one effort decreases the need for the other effort, while multiplicative efforts can be interpreted as complementary, so that employing more of one effort increases the need for the other effort.[3]

One example of additive efforts is competition for privileges which are such that many different kinds of efforts are possible to obtain the privileges. The rent seeker may apply its various strengths, which all add up to the total rent seeking effort, and obtain the privileges in multifarious

[1] Group contests have been analyzed by many, e.g. Katz and Tokatlidu (1996), Skaperdas and Syropoulos (1997), Muller and Warneryd (2001), Hausken (2005, 2012), Baik (2008), Münster (2009), surveyed early by Nitzan (1994) and more recently by Konrad (2009, chapters 5.5 and 6).

[2] When also accounting for actors incurring unit costs of efforts operating in the contest success function with additive efforts, Hausken (2016) shows (Proposition 3) that "multiple efforts with different production functions and unit effort costs can be of the same kind or nature, can be substitutes for each other, or can complement each other in various ways."

[3] More precisely, "X and Y are substitutes if the demand for X increases when the price of Y increases," http://www.investopedia.com/terms/s/substitute.asp . Further, "the complementary good has little to no value when consumed alone but, when combined with another good or service, it adds to the overall value of the offering," http://www.investopedia.com/terms/c/complement.asp?ad=dirN&qo=investopediaSiteSearch&qsrc=0&o=40186 , retrieved September 11, 2016

ways. One example of multiplicative efforts is competition for R&D budgets where multiple efforts have to be jointly present, e.g. focus on scientific value, implementation, and impact. See Hausken (2016) for further examples of additive and multiplicative efforts, and Arbatskaya and Mialon (2010) for examples of multiplicative efforts.

Let us position the paper more broadly within the literature. Rai and Sarin (2009) generalize axiomatizations to allow multiple types of investments. They exemplify with the Cobb-Douglas production function and a linear production function with two additive efforts, where one investment is fixed, as analyzed by e.g. Nti (2004) and Hausken and Zhuang (2012). Epstein and Hefeker (2003) consider rent seeking reinforced multiplicatively with a second effort. Combined rent seeking and sabotage efforts has been analyzed by Konrad (2000), Chen (2003), Krakel (2005), Amegashie and Runkel (2007), surveyed by Chowdhury and Gürtler (2015). Multiple efforts, i.e. production and appropriation, have been analyzed by Hirshleifer (1995), Skaperdas and Syropoulos (1997), and Hausken (2005), where contest success depends only on appropriation. Finally, Chowdhury and Sheremeta (2015) propose a procedure to identify strategically equivalent contests which generate the same equilibrium efforts but different equilibrium payoffs.

Section 2 specifies which of Münster's (2009) nine axioms apply in this paper, which additional axioms apply generally, which axioms apply for the additive production function, and which axioms apply for the Cobb-Douglas production function. Section 3 presents results. Section 4 concludes.

2 Axiomatization

Consider G groups denoted as $\Gamma = \{1, \ldots, G\}$, $m_g \geq 1$ members in group g, and $n = \sum_{g=1}^{G} m_g$ players. Player i in group g exerts effort $x_{ig} \in \mathbb{R}_+$. Define $\boldsymbol{x}_g = \left(x_{1g}, \ldots, x_{m_g g}\right)$ as the vector of efforts by the members of group g, $\boldsymbol{x} = (\boldsymbol{x_1}, \ldots, \boldsymbol{x_G})$ as the vector of efforts by all players, $\boldsymbol{x}_{-g}$ as the vector of efforts by all players not belonging to group g, and $\boldsymbol{x}_{-ig}$ as the vector of efforts by all players except player i in group g. A contest success function $p_g(\boldsymbol{x})$, $p_g: \mathbb{R}_+^n \to \mathbb{R}_+$ expresses for any group $g \in \Gamma$ the probability that group g wins the contest, or the share of a rent that group g gets.

2.1 General axioms

Münster (2009) presents the following nine axioms.

Axiom 1. $\sum_{g=1}^{G} p_g(\boldsymbol{x}) = 1$ and $p_g(\boldsymbol{x}) \geq 0$ for all $g \in \Gamma$.

Axiom 2. For all $g \in \Gamma$ and $i \in \{1, \dots, m_g\}$: if $\hat{x}_{ig} > x_{ig}$, then

(i) $p_g(\hat{x}_{ig}, \boldsymbol{x}_{-ig}) \geq p_g(x_{ig}, \boldsymbol{x}_{-ig})$, with strict inequality unless $p_g(x_{ig}, \boldsymbol{x}_{-ig}) = 1$, and

(ii) for all $k \neq g, k \in \Gamma$: $p_k(\hat{x}_{ig}, \boldsymbol{x}_{-ig}) \leq p_k(x_{ig}, \boldsymbol{x}_{-ig})$.

Axiom 3. A contest success function satisfies between-group-anonymity if, whenever $m_g = m_k$, $p_g(\boldsymbol{x}_g, \boldsymbol{x}_k, \boldsymbol{x}_{\Gamma\setminus\{g,k\}}) = p_g(\boldsymbol{x}_k, \boldsymbol{x}_g, \boldsymbol{x}_{\Gamma\setminus\{g,k\}})$.

Axiom 3'. A contest success function satisfies within-group-anonymity if, for any group $g \in \Gamma$ and for any bijection $\psi: \{1, \dots, m_g\} \to \{1, \dots, m_g\}$,

$$p_g(\boldsymbol{x}_g, \boldsymbol{x}_{-g}) = p_g(\hat{\boldsymbol{x}}_g, \boldsymbol{x}_{-g})$$

where $\hat{\boldsymbol{x}}_g = \left(x_{\psi(1)g}, \dots, x_{\psi(m_g)g}\right)$ is the vector of efforts of the members of group g after a permutation according to ψ.

Axiom 4. Let $p_g^M(\boldsymbol{x})$ be group g's probability of winning a subcontest played by a subset $M \subset \Gamma$ consisting of at least two groups. Then for all $g \in M$,

$$p_g^M(\boldsymbol{x}) = \frac{p_g(\boldsymbol{x})}{\sum_{k \in M} p_k(\boldsymbol{x})} \; \forall \boldsymbol{x} \; \boldsymbol{s.t.} \; \boldsymbol{x}_M \neq \boldsymbol{0}.$$

Axiom 5. $p_g^M(\boldsymbol{x})$ is independent of the efforts of players belonging to groups not in M.

Axiom 6. For all $\lambda > 0$ and all $g \in \Gamma$, $p_g(\lambda \boldsymbol{x}) = p_g(\boldsymbol{x})$.

Axiom 7. For all $\lambda > 0$ and all $g \in \Gamma$, $p_g(\boldsymbol{x} + \lambda \boldsymbol{1}) = p_g(\boldsymbol{x})$, where $\boldsymbol{1} = (1, \dots, 1)$.

Axiom 8. Fix any $\Delta > 0$ such that $\Delta \leq x_{ig}$ for all $i \in \{1, \dots, m_g\}$. Define

$$\hat{\boldsymbol{x}}_g^{ij} = \left(x_{1g}, \dots, x_{(i-1)g}, x_{ig} - \Delta, x_{(i+1)g}, \dots, x_{(j-1)g}, x_{jg} + \Delta, x_{(j+1)g}, \dots, x_{m_g g}\right)$$

Then

$$p_g\left(\boldsymbol{x}_g^{ij}, \boldsymbol{x}_{-g}\right) = p_g\left(\widehat{\boldsymbol{x}}_g^{ij}, \boldsymbol{x}_{-g}\right)$$

for all $i, j \in \{1, \dots, m_g\}$ and all $g \in \Gamma$.

This paper assumes the general validity of Axioms 1,2,4,5,6. Axioms 3 and 3' about between-group and within-group-anonymity are not adopted since they restrictively assume the same effort conversion technology for all groups, and for all players. Instead we assume the following weaker anonymity axioms, analogous to Rai and Sarin's (2009) Axiom 6 and Hausken's (2016) Axiom 8.

Axiom 3''. A contest success function satisfies between-group-anonymity if, whenever $m_g = m_k$,
$p_g\left(\boldsymbol{x}_g, \boldsymbol{x}_k, \boldsymbol{x}_{\Gamma\backslash\{g,k\}}\right) = p_k\left(\boldsymbol{x}_g, \boldsymbol{x}_k, \boldsymbol{x}_{\Gamma\backslash\{g,k\}}\right)$ when $\boldsymbol{x}_g = \boldsymbol{x}_k$.

Axiom 3'''. A contest success function satisfies within-group-anonymity if, for any group $g \in \Gamma$ and for any bijection $\psi: \{1, \dots, m_g\} \rightarrow \{1, \dots, m_g\}$,
$p_g\left(\left(x_{1g}, \dots, x_{m_g g}\right), \boldsymbol{x}_{-g}\right) = p_g\left(\left(x_{\psi(1)g}, \dots, x_{\psi(m_g)g}\right), \boldsymbol{x}_{-g}\right)$ when $x_{ig} = x_{\psi(i)g}$ $\forall i = 1, \dots, m_g$,
where $\left(x_{\psi(1)g}, \dots, x_{\psi(m_g)g}\right)$ is the vector of efforts of the members of group g after a permutation according to ψ.

Axiom 6 states that the success probabilities are unaffected by all players altering their efforts by a fixed multiplicative constant λ. In contrast, Axiom 7 assumes that the success probabilities are unaffected by all players adding a constant λ to their efforts. Axiom 7 supports Hirshleifer's (1989) logistic contest success function, where only the difference between the groups' or players' efforts matters. Münster's (2009) Axiom 8 is not generally adopted. Hausken (2016) shows that Axiom 8 is valid for additive efforts if multiple efforts by a player are substitutable with equivalent impact in the contest success function, and that Axiom 8 is invalid for the Cobb-Douglas production function.

We introduce the following axiom corresponding to Hausken's (2016) Axiom 1 (iv):

Axiom 9. For all $g \in \Gamma$, if $\boldsymbol{x}_g = \boldsymbol{0}$ and $\boldsymbol{x}_{-g} \in \mathbb{R}_+^{\sum_{k=1,k\neq g}^{G+1} m_g} \neq (\boldsymbol{0}, \dots, \boldsymbol{0})$, then $p_g(\boldsymbol{x}) = 0$.

Axiom 9 states that if group g exerts no efforts, and at least one other group exerts at least one strictly positive effort, then group $g's$ winning probability is 0.

Axiom 1 corresponds to Arbatskaya and Mialon's (2010) Axiom 1 (i), with the minor difference that they assume $\sum_{g=1}^{G} p_g(\boldsymbol{x}) \leq 1$. Axiom 9 corresponds to Arbatskaya and Mialon's (2010) Axiom 1 (iii), except that they do not require $\mathbb{R}_+^{\sum_{k=1,k\neq g}^{G+1} m_g} \neq (\boldsymbol{0}, \dots, \boldsymbol{0})$. Corresponding to Hausken's (2016) Axiom 1 (vi) we require $\mathbb{R}_+^{\sum_{k=1,k\neq g}^{G+1} m_g} \neq (\boldsymbol{0}, \dots, \boldsymbol{0})$ to enable the following axiom:

Axiom 10. If $\boldsymbol{x} = (\boldsymbol{0}, \dots, \boldsymbol{0})$, then $p_1(\boldsymbol{x}) = p_2(\boldsymbol{x}) = \cdots = p_G(\boldsymbol{x})$ and $\sum_{g=1}^{G} p_g(\boldsymbol{x}) = 1$.

Axiom 10 assumes that if no groups exert efforts, then their winning probabilities are equal and sum to one.

Axiom 11 corresponds to Arbatskaya and Mialon's (2010) Axiom 2 and Hausken's (2016) Axiom 2, and is a version of the independence from irrelevant alternatives.

Axiom 11. For all $g \neq k \neq s \in \Gamma$, the odds ratio $\frac{p_g(\boldsymbol{x})}{p_k(\boldsymbol{x})}$ does not depend on $\boldsymbol{x}_s$ for $\boldsymbol{x}_g \in \mathbb{R}_+^{m_g}$, $\boldsymbol{x}_k \in \mathbb{R}_{++}^{m_g}$, and $\boldsymbol{x}_s \in \mathbb{R}_+^{m_g}$.

Axiom 12 corresponds to Arbatskaya and Mialon's (2010) and Hausken's (2016) Axiom 3. It expresses that if group g is inactive, then the contest reduces to a player contest among $G - 1$ groups.

Axiom 12. For all $g \in \Gamma$ and $\boldsymbol{x} = (\boldsymbol{x_1}, \dots, \boldsymbol{x_G}) \in \mathbb{R}_+^{\sum_{g=1}^{G} m_g}$, $p_g(\boldsymbol{x_{-g}}) = p_h(\boldsymbol{x_{-g}}, \boldsymbol{x_g} = \boldsymbol{0})$.

Axiom 13. For all $g \in \Gamma$ and $\lambda > 0$, $p_g\big(\lambda f_1(\boldsymbol{x_1}), \lambda f_2(\boldsymbol{x_2}), \dots, \lambda f_G(\boldsymbol{x_G})\big) = p_g\big(f_1(\boldsymbol{x_1}), f_2(\boldsymbol{x_2}), \dots, f_G(\boldsymbol{x_G})\big) = p_g(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_G}) = p_g(\lambda \boldsymbol{x_1}, \lambda \boldsymbol{x_2}, \dots, \lambda \boldsymbol{x_G})$.

Axiom 13 corresponds to Hausken's (2016) Axiom 5. It is stronger than the homogeneity Axiom 6 since it multiplies λ with all groups' production functions and all groups' efforts. It is weaker than Arbatskaya and Mialon's (2010) homogeneity Axiom 5 adapted for groups below as Axiom 20CD to support the Cobb-Douglas production functions.

Axiom 14. For all $g \in \Gamma$, if $a \in \mathbb{R}^1_{++}$ or $b > 1$, then $p_h\left(f_1(\boldsymbol{x_1}), \dots, f_g(\boldsymbol{x_g})b + a, \dots, f_G(\boldsymbol{x_G})\right) > p_h\left(f_1(\boldsymbol{x_1}), \dots, f_g(\boldsymbol{x_g}), \dots, f_G(\boldsymbol{x_G})\right)$.

Axiom 14 corresponds to Hausken's (2016) Axiom 7. It states that adding a strictly positive constant to a group's production function, or multiplying a group's production function with a constant strictly larger than one, causes higher success probability.

2.2 Axioms for the additive production function

Axiom 15a. For all $g \in \Gamma$, if $x_{ig} \in \mathbb{R}^1_{++}$ for at least one $i \in \{1, \dots, m_g\}$ and $\boldsymbol{x}_{-ig} \in \mathbb{R}_+^{\sum_{g=1}^{G} m_g - 1}$, then $p_g(\boldsymbol{x}) > 0$.

Axiom 15a corresponds to Hausken's (2016) Axiom 1 (ii), but applied to groups. It states that if player i in group g exerts strictly positive effort, then group g's winning probability is strictly positive regardless of the other players' efforts in all G groups.

Axiom 16a. For all $g \in \Gamma$, if $x_{ig} \in \mathbb{R}^1_{++}$ for at least one $i \in \{1, \dots, m_g\}$ and $\boldsymbol{x}_{-g} = (\boldsymbol{0}, \dots, \boldsymbol{0})$, then $p_g(\boldsymbol{x}) = 1$ and $p_g(\boldsymbol{x}_{-g}) = 0$.

Axiom 16a, corresponding to Hausken's (2016) Axiom 1 (iii), states that if player i in group g exerts strictly positive effort, and all other groups exert no efforts, then group g's winning probability is 1 and the other groups' winning probabilities are 0.

Axiom 17a. If $x_{ig} \in \mathbb{R}^1_{++}$ for at least one $i \in \{1, \dots, m_g\}$ and $\boldsymbol{x}_{-ig} \in \mathbb{R}_+^{\sum_{g=1}^G m_g - 1}$ for some $g \in \Gamma$, then $\sum_{g=1}^G p_g(\boldsymbol{x}) = 1$.

Axiom 17a, corresponding to Hausken's (2016) Axiom 1 (v), states that if at least one player exerts at least one strictly positive effort, then the winning probabilities sum to one.

Axiom 18a. For all $g \in \Gamma$ and $i \in \{1, \dots, m_g\}$, $f_g(\boldsymbol{x_g})$ is separable into two functions $f_g(\boldsymbol{x}_{-ig})$ and $f_g(x_{ig})$ so that $\partial f_g(\boldsymbol{x_g})/\partial x_{ig} = \partial f_h(x_{ig})/\partial x_{ig} \neq 0$ and $\partial f_h(\boldsymbol{x}_{-ig})/\partial x_{ig} = 0$.

Axiom 18a, corresponding to Hausken's (2016) Axiom 9, assumes separability of $f_g(\boldsymbol{x_g})$ into one function $f_g(\boldsymbol{x}_{-ig})$ dependent on $m_g - 1$ efforts by group g, and one function $f_g(x_{ig})$ dependent only on effort x_{ig} by player i in group g. Differentiation of the unseparated function $f_g(\boldsymbol{x_g})$ and the separated function $f_g(x_{ig})$ with respect to effort x_{ig} cause the same result. Separability prevents $f_g(\boldsymbol{x_g})$ from having e.g. multiplicative terms between effort x_{ig} and at least one other effort x_{jg}, $j \in \{1, \dots, m_g\}$, $j \neq i$. Axiom 18a does not support the multiplicative Cobb-Douglas production function.

Axiom 19a. Fix any $\Delta > 0$ such that $\Delta \leq x_{ig}$ for all $i \in \{1, \dots, m_g\}$. Define

$$\widehat{\boldsymbol{x}}_g^{ij} = \left(x_{1g}, \dots, x_{(i-1)g}, x_{ig} - \Delta, x_{(i+1)g}, \dots, x_{(j-1)g}, x_{jg} + \Delta, x_{(j+1)g}, \dots, x_{m_g g}\right) \quad (1)$$

Then

$$p_g(\boldsymbol{x}_g^{ij}, \boldsymbol{x}_{-g}) = p_g(\widehat{\boldsymbol{x}}_g^{ij}, \boldsymbol{x}_{-g}) \quad (2)$$

assuming that efforts x_{ig} and x_{jg} with equivalent impact in the contest success function are substitutable for all $i, j \in \{1, \dots, m_g\}$ and all $g \in \Gamma$ while preserving $\Delta > 0$.

Axiom 19a is a special case of Münster's (2009) Axiom 8 and corresponds to Hausken's (2016) Axiom 10. It states that if two efforts x_{ig} and x_{jg} by players i and j are substitutable with equivalent impact in the contest success function, then subtracting Δ from x_{ig} and adding Δ to x_{jg} does not affect the winning probabilities. Hausken (2016) shows that Axiom 19a does not support the multiplicative Cobb-Douglas production function.

2.3 Axioms for the Cobb-Douglas production function

Axiom 15CD. For all $g \in \Gamma$, if $\boldsymbol{x}_g \in \mathbb{R}_{++}^{m_g}$ and $\boldsymbol{x}_{-g} \in \mathbb{R}_{+}^{\sum_{k=1,k\neq g}^{G} m_g}$, then $p_g(\boldsymbol{x}) > 0$.

Axiom 15CD contrasts with Axiom 15a and corresponds to Arbatskaya and Mialon's (2010) Axiom 1 (ii), but applied to groups. It requires that all m_g group members in group g exert strictly positive effort in order for group g's winning probability to be strictly positive regardless of the other groups' efforts.

Axiom 16CD. For all $g \in \Gamma$, if $\boldsymbol{x}_g \in \mathbb{R}_{++}^{m_g}$ and $\boldsymbol{x}_{-g} = (\boldsymbol{0}, \dots, \boldsymbol{0})$, then $p_g(\boldsymbol{x}) = 1$ and $p_g(\boldsymbol{x}_{-g}) = 0$.

Axiom 16CD contrasts with Axiom 16a and states that if all m_g group members in group g exert strictly positive effort and all group members in all other groups exert no effort, then group g's winning probability is 1 and the other groups' winning probabilities are 0.

Axiom 17CD. If $\boldsymbol{x}_g \in \mathbb{R}_{++}^{m_g}$ and $\boldsymbol{x}_{-g} \in \mathbb{R}_{+}^{\sum_{k=1,k\neq g}^{G} m_g}$ for some $g \in \Gamma$, then $\sum_{g=1}^{G} p_g(\boldsymbol{x}) = 1$.

Axiom 17CD contrasts with Axiom 17a and corresponds to Arbatskaya and Mialon's (2010) Axiom 1 (iv), but applied to groups. It requires that all m_g group members in group g exert strictly positive effort in order for the winning probabilities to sum to one.

Arbatskaya and Mialon (2010) consider two players where each player exerts equally many efforts K. They do that in order to match efforts by the two players against each other. Analogously in this paper, we match different players in different groups against each other. To adopt their Axiom 5 to groups we assume that each group has as many members as the largest group, i.e. $K = \max_{g=1,\dots,G} m_g$. (Hausken (2016) makes an analogous assumption to enable different players to exert different numbers of efforts. The superfluous efforts for the player exerting fewest efforts are set to zero.) This means that group g where $m_g < K$ has at least one dummy member which is matched against the last member in the largest group.

Axiom 20CD. For all $g \in \Gamma$, $i \in \{1, \dots, K\}$ where $K = \max_{g=1,\dots,G} m_g$, and $\lambda > 0$,

$$p_g(x_{11}, \dots, \lambda x_{i1}, \dots, x_{K1}, x_{12}, \dots, \lambda x_{i2}, \dots, x_{K2}, \dots, x_{1G}, \dots, \lambda x_{i1}, \dots, x_{KG}) = p_g(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\boldsymbol{G}})$$

if $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\boldsymbol{G}}) \in \mathbb{R}_+^{GK}$

Axiom 20CD corresponds to Arbatskaya and Mialon's (2010) homogeneity Axiom 5. It is stronger than the homogeneity Axiom 6 which multiplies λ with all players' efforts in all groups, and different and in some sense stronger than Axiom 13. Axiom 20CD matches members in all groups against each other and states that an equiproportionate change in the matched group member's efforts does not affect players' success probabilities. This kind of homogeneity is invariant to changes in units of measurement for the efforts of the matched players.

2.4 Axiomatic overview

Table 1 specifies how the axioms in this paper correspond to axioms in earlier papers. Empty cells means that axioms do not correspond to axioms in the earlier papers. The sign ≈ means that the axioms are approximately equal. Some of these are discussed above.

Table 1 How the axioms in this paper correspond to axioms in earlier papers.

	Münster (2009)	Hausken (2016)	Arbatskaya and Mialon (2010)
Axiom 1	Equivalent	≈Axiom 1 (i)	≈Axiom 1 (i)
Axiom 2	Equivalent	≈Axiom 4	≈Axiom 4
Axiom 3	Equivalent		Axiom 7
Axiom 3’	Equivalent		Axiom 7
Axiom 3’’		Axiom 8	
Axiom 3’’’		Axiom 8	
Axiom 4	Equivalent		
Axiom 5	Equivalent		
Axiom 6	Equivalent	≈Axiom 5	
Axiom 7	Equivalent	Axiom 6	
Axiom 8	Equivalent	≈Axiom 10	
Axiom 9		Axiom 1 (iv)	≈Axiom 1 (iii)
Axiom 10		Axiom 1 (vi)	
Axiom 11		Axiom 2	Axiom 2
Axiom 12		Axiom 3	Axiom 3
Axiom 13		Axiom 5	Axiom 5
Axiom 14		Axiom 7	
Axiom 15a		Axiom 1 (ii)	
Axiom 16a		Axiom 1 (iii)	
Axiom 17a		Axiom 1 (v)	
Axiom 18a		Axiom 9	
Axiom 19a	≈Axiom 8	Axiom 10	
Axiom 15CD			Axiom 1 (ii)
Axiom 16CD			
Axiom 17CD			Axiom 1 (iv)
Axiom 20CD			Axiom 5

3 Results

Münster (2009) proves the following two theorems, also valid in this paper.

Theorem 1. Assume that Axioms 1,2,4,5 hold, $M \subset \Gamma$, $G \geq 2$. Then, for each $g \in M$, a non-negative and strongly increasing function $f_g\colon \mathbb{R}_+^{m_g} \to \mathbb{R}_+$ exists such that

$$p_g^M(\boldsymbol{x}) = \frac{f_g(\boldsymbol{x}_g)}{\sum_{k \in M} f_k(\boldsymbol{x}_k)} \ \forall \boldsymbol{x}\ \boldsymbol{s.t.}\ \boldsymbol{x}_M \neq \boldsymbol{0} \tag{3}$$

Theorem 2. If the contest success function satisfies Axioms 1,2,4,5,6, then it satisfies (3) and the production functions f_k are homogeneous of the same degree $r > 0$.

3.1 Results for the additive production function

Lemma 1a. If Axioms 1,9,10,11,12,15a,16a,17a hold, then group g's contest success function, $g \in \Gamma$, in a G-group contest where each group g exerts m_g efforts is

$$p_g(\boldsymbol{x}) \begin{cases} = \dfrac{f_g(\boldsymbol{x_g})}{\sum_{k\in\Gamma} f_k(\boldsymbol{x_k})} \; if \; x_{is} \in \mathbb{R}^1_{++} \text{ for at least one } i \in \{1, \dots, m_s\} \\ \qquad\qquad \text{in at least one group } s \in \Gamma \\ \geq 0 \text{ and } \sum_{k\in\Gamma} p_k(\boldsymbol{x}) \leq 1 \; if \; x_{is} \in \mathbb{R}^1_+ \backslash \mathbb{R}^1_{++} = \{0\} \text{ for at} \\ \qquad\qquad \text{least one } i \in \{1, \dots, m_s\} \; \forall s \in \Gamma \\ = 0 \; if \; \boldsymbol{x_g} = \boldsymbol{0} \text{ and } \boldsymbol{x_{-g}} \in \mathbb{R}_+^{\sum_{k=1,k\neq g}^{G} m_g} \text{ and } \boldsymbol{x_{-g}} \neq (\boldsymbol{0}, \dots, \boldsymbol{0}) \\ = 1/G \; if \; \boldsymbol{x} = (\boldsymbol{0}, \dots, \boldsymbol{0}) \end{cases} \tag{4}$$

where group $g's$ production function $f_g(\boldsymbol{x_g})$ satisfies

$$f_g(\boldsymbol{x_g}) \begin{cases} > 0 \; if \; x_{ig} \in \mathbb{R}^1_{++} \text{ for at least one } i \in \{1, \dots, m_g\} \\ \geq 0 \; if \; x_{ig} \in \mathbb{R}^1_{+} \text{ for at least one } i \in \{1, \dots, m_g\} \\ = 0 \; if \; \boldsymbol{x_g} = \boldsymbol{0} \end{cases} \tag{5}$$

Proof. Appendix A.

Lemma 1a corresponds to Hausken's (2016) Lemma 1, adopted to groups. Essential for Lemma 1a is that (strict) positivity for one of group g's m_g efforts is sufficient to cause $f_g(\boldsymbol{x_g})$ and $p_g(\boldsymbol{x})$ to be (strictly) positive.

The redundant second line in (4) is included to illustrate the contrast to Lemma 1CD and Arbatskaya and Mialon (2010). They, and correspondingly Lemma 1CD, require all group $s's$ m_s efforts to be strictly positive, giving $\mathbb{R}^{m_g}_{+} \backslash \mathbb{R}^{m_g}_{++}$ which is a collection of coordinate hyperplanes (in the positive orthant), whereas we require at least one of group $s's$ m_s efforts to be strictly positive effort which gives the singleton $\mathbb{R}^1_+ \backslash \mathbb{R}^1_{++} = \{0\}$.

Lemma 2a. Assume that Axioms 1,2, 3'',3''',6,9,10,11,12,13,14,15a,16a,17a,18a,19a hold. Then the additive production function

$$f_g(\boldsymbol{x_g}) = \left(\sum_{i=1}^{m_g} \delta_{ig} {x_{ig}}^{\mu_{ig}} \right)^r \tag{6}$$

satisfies Lemma 1a, where $\delta_{ig} \geq 0$, $\mu_{ig} \geq 0$, and $r \geq 0$.

Proof. Appendix B.

Lemma 2a corresponds to Hausken's (2016) Lemma 2, adopted to groups. Axiom 19a restrictively supports Lemma 2a only for the special case that $\delta_{ig} = \delta_g$ and $\mu_{ig} = 1$ for all $g \in \Gamma$ and $i \in \{1, \dots, m_g\}$. For the more general case that $\delta_{ig} \neq \delta_g$ and μ_{ig} differs from 1, support by the other axioms is needed. Münster's (2009) Proposition 2 assumes the more general Axiom 8, and thus not Axiom 19a, causing a special case of Lemma 2 where $\delta_{ig} = \delta_g$ and $\mu_{ig} = 1$ $\forall i = 1, \dots, m_g$.

Lemma 3a. Assume that Axioms 1,2, 3'',3''',7,9,10,11,12,13,14,15a,16a,17a,18a,19a hold. Then the additive production function

$$f_g(\boldsymbol{x_g}) = a_g Exp\left(r \sum_{i=1}^{m_g} \delta_{ig} {x_{ig}}^{\mu_{ig}}\right) \tag{7}$$

satisfies Lemma 1a, where $\delta_{ig} \geq 0$, $\mu_{ig} \geq 0$, $a_g \geq 0$, and $r \geq 0$.

Proof. Follows from the proof of Lemma 2a, replacing Axiom 6 with Axiom 7. The proof is as Münster's (2009) proof of his Proposition 3, which analogously assumes Axiom 7.

Lemma 3a corresponds to Hausken's (2016) Lemma 3, adopted to groups. Münster's (2009) Proposition 3 assumes the more general Axiom 8, and thus not Axiom 19a. Whereas Lemma 2a based on Axiom 6 supports the ratio form contest success function, Lemma 3a based on Axiom 7 supports the logistic contest success function (Hirshleifer, 1989).

3.2 Results for the Cobb-Douglas production function

Lemma 1CD. If Axioms 1,9,11,12,15CD,16CD,17CD hold, then group g's contest success function, $g \in \Gamma$, in a G-group contest where each group g exerts m_g efforts is

$$p_g(\boldsymbol{x})\begin{cases}=\dfrac{f_g(\boldsymbol{x_g})}{\sum_{k\in\Gamma} f_k(\boldsymbol{x_k})}\ if\ \boldsymbol{x_s}\in\mathbb{R}^1_{++}\ \text{for at least one group}\ s\in\Gamma\\ \geq 0\ \text{and}\ \sum_{k\in\Gamma} p_k(\boldsymbol{x})\leq 1\ if\ \boldsymbol{x_s}\in\mathbb{R}^{m_k}_{+}\backslash\mathbb{R}^{m_k}_{++}\ \forall s\in\Gamma\\ =0\ if\ \boldsymbol{x_g}=\boldsymbol{0}\ \text{and}\ \boldsymbol{x_{-g}}\in\mathbb{R}_+^{\sum_{k=1,k\neq g}^{G} m_g}\ \text{and}\ \boldsymbol{x_{-g}}\neq(\boldsymbol{0},\dots,\boldsymbol{0})\\ =1/G\ if\ \boldsymbol{x}=(\boldsymbol{0},\dots,\boldsymbol{0})\end{cases}\tag{8}$$

where group $g's$ production function $f_g(\boldsymbol{x_g})$ satisfies

$$f_g(\boldsymbol{x_g})\begin{cases}>0\ if\ \boldsymbol{x_g}\in\mathbb{R}^{m_g}_{++}\\ \geq 0\ if\ \boldsymbol{x_g}\in\mathbb{R}^{m_g}_{+}\\ =0\ if\ \boldsymbol{x_g}=\boldsymbol{0}\end{cases}\tag{9}$$

Proof. The proof corresponds to that provided by Arbatskaya and Mialon (2010) for their Lemma 1, adopted straightforwardly to groups.

Lemma 1CD corresponds to Arbatskaya and Mialon's (2010) Lemma 1, adopted to groups.

Lemma 2CD. Assume that 1,2,6,9,11,12,15CD,16CD,17CD,20CD hold. Then the Cobb-Douglas production function

$$f_g(\boldsymbol{x_g})=\delta_g\left(\prod_{i=1}^{m_g} {x_{ig}}^{\mu_{ig}}\right)^r\tag{10}$$

satisfies Lemma 1CD, where $\delta_g\geq 0$, $\mu_{ig}\geq 0$, and $r\geq 0$.

Proof. The proof corresponds to that provided by Arbatskaya and Mialon (2010) for their Proposition 1, adopted straightforwardly to groups. Raising to the exponent r is supported by Axiom 6, as proved in Appendix B for Lemma 2a.

Lemma 2CD corresponds to Arbatskaya and Mialon's (2010) Proposition 1, adopted to groups, and raised to the exponent r for increased generality.

Lemma 3CD. Assume that 1,2,7,9,11,12,15CD,16CD,17CD,20CD hold. Then the Cobb-Douglas production function

$$f_g(\boldsymbol{x_g}) = \delta_g Exp\left(r\prod_{i=1}^{m_g} x_{ig}{}^{\mu_{ig}}\right) \tag{11}$$

satisfies Lemma 1CD, where $\delta_g \geq 0$, $\mu_{ig} \geq 0$, and $r \geq 0$.

Proof. Just as the proof of Lemma 3a follows from the proof of Lemma 2a by replacing Axiom 6 with Axiom 7, the proof of Lemma 3CD follows from the proof of Lemma 2CD by replacing Axiom 6 with Axiom 7, applying Münster's (2009) proof of his Proposition 3.

Whereas Lemma 2CD based on Axiom 6 supports the ratio form contest success function, Lemma 3CD based on Axiom 7 supports the logistic contest success function (Hirshleifer, 1989).

4 Conclusion

This paper axiomatizes group contest success functions for general production functions accounting for addition, exponentiation, and multiplication, as expressed by the additive production function and the Cobb-Douglas production function. Earlier axiomatizations are developed further, in particular Münster's (2009) axiomatization of group contest success functions, Hausken's (2016) axiomatization of contests with additive production functions, and Arbatskaya and Mialon's (2010) axiomatization of contests with the Cobb-Douglas production function. Additive efforts can be interpreted as substitutable, whereas multiplicative efforts can be interpreted as complementary.

The distinguishing feature for axioms supporting the additive production function is that it is sufficient that one member of a group exerts effort to ensure that the group as a whole exerts effort. That is, all members' efforts are added according to each member's characteristics, and contribute to the group effort also when only some members, or all members except one, exert no effort. In contrast, the distinguishing feature for axioms supporting the Cobb-Douglas production function is that all group members need to exert efforts to ensure that the group as a whole exerts effort.

Further, substitutable additive efforts with equivalent impact in the contest success function are supported by an axiom where adding an amount to the first effort and subtracting the same amount from the second effort does not change the winning probabilities. Such an axiom does not support the Cobb-Douglas production function. In contrast, the Cobb-Douglas production function is supported by a strong homogeneity axiom where an equiproportionate change in matched group member's efforts does not affect groups' winning probabilities. Such an axiom does not support the additive production function.

Future research should determine Nash equilibria and rent dissipation for groups with different characteristics, matched against empirical measurements of the nature and prevalence of additive and multiplicative production functions in groups' rent seeking.

Appendix A Proof of Lemma 1a

The proof follows Arbatskaya and Mialon's (2010) template, also adopted by Hausken (2016), and accounting for the different axioms. For all $g \neq k \neq s \in \Gamma$ assume $x_{ik} \in \mathbb{R}^1_{++}$ for at least one $i \in \{1, \dots, m_k\}$ and $\boldsymbol{x_{-k}} \in \mathbb{R}^{2K}_+$, which without loss of generality means that group k is the group exerting at least one positive effort, i.e. by player i. Axioms 15a and 16a imply $p_k(\boldsymbol{x}) > 0$. Axiom 11 implies that $\frac{p_g(\boldsymbol{x})}{p_k(\boldsymbol{x})}$ does not depend on $\boldsymbol{x_s}$. Hence Arbatskaya and Mialon's (2010) subsequent equations apply also for the different axioms, causing Lemma 1a.

Appendix B Proof of Lemma 2a

The proof follows Hausken's (2016) template for his Lemma 2. Axioms 15a and 16a support Lemma 2a since $x_{ig} > 0$ implies $f_g(\boldsymbol{x_g}) > 0$ and $p_g(\boldsymbol{x}) > 0$. Axiom 17a supports Lemma 2a for the same reason, and thus $\sum_{g \in \Gamma} p_g(\boldsymbol{x}) = 1$. Axioms 1,9,10,11,12 obviously support Lemma 2a. Axiom 4 supports Lemma 2a since (6) in Lemma 2a inserted into Lemma 1a gives $\partial p_g(\boldsymbol{x})/\partial x_{ig} \geq 0$, and hence $p_g(\boldsymbol{x})$ is nondecreasing in x_{ig}. Axioms 6,13,14,3'',3'''obviously support Lemma 2a. Axiom 18a supports Lemma 2a since separating (6) for $r = 1$ gives $f_g(\boldsymbol{x_g}) = f_g(x_{ig}) + f_g(\boldsymbol{x_{-ig}}) = \delta_{ig} {x_{ig}}^{\mu_{ig}} + \sum_{j=1, j \neq i}^{m_g} \delta_{jg} {x_{jg}}^{\mu_{jg}}$ and differentiation gives $\frac{\partial f_g(x_g)}{\partial x_{ig}} = \frac{\partial f_g(x_{ig})}{\partial x_{ig}} = \delta_{ig} \mu_{ig} {x_{ig}}^{\mu_{ig}-1} \neq 0$ and $\frac{\partial f_g(\boldsymbol{x_{-ig}})}{\partial y_{hk}} = 0$. Axiom 6 supports raising the sum in (6) to the exponent r since, as Carter (2001, p. 351) points out, a homogeneous function of one variable is a multiple of a power function. The exponent r corresponds to Münster's (2009) generalization from Proposition 1 to Proposition 2. To show that Axiom 19a supports Lemma 2a, the proof is as in Hausken (2016).

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