Derivatives and Risk Management in Commodity Markets

Topic 3: Option pricing

Bård Misund, Professor of Finance
University of Stavanger
uis.no

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About the author

Social Media: LinkedIn, Google Scholar, Twitter, YouTube, Vimeo, Facebook
Employee Pages: UiS (Nor), UiS (Eng), NORCE1, NORCE2
Personal Webpages: bardmisund.com, bardmisund.no, UiS
Research: Researchgate, IdeasRePEc, Publons, Orcid, SSRN,
Others: Encyclopedia, Cristin1, Cristin2, Brage UiS, Brage USN, Nofima, FFI, SNF, HHUiS WP, Risk Net
Op-Eds: Nationen, Fiskeribladet, Nordnorsk Debatt

Topics

- Upper & lower bounds for options
- The put-call parity
- Early exercise
- Option pricing using the binomial model
- Option pricing using the trinomial model
- Option pricing using the Black-Scholes model

Learning objectives: Upper & lower bounds, put-call parity & Early exercise

- Know how to derive upper and lower bounds for European calls and puts
- Know what the put-call parity is and how we can derive it
- Know why it is never optimal to exercise an American call before maturity
- Know why it is always optimal to exercise an American put before maturity (as long as it is sufficiently in the money)

Learning objectives: applyting the 1-step binomial tree

- Be able to derive the 1-step binomial pricing formula using:
  - The «delta hedging» approach
  - The «replicating portfolio» approach

- Be able to identify arbitrage opportunities and devise strategies to take advantage of arbitrage opportunities (Hint: «Buy low, sell high»)

- Be able to value options using multi-step models (>1 step)

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Learning objectives: american options & trinomial trees

- Be able to use the binomial model to price American options (value of early exercise)
- Be able to use the binomial model to price exotic options
- Be able to price options using trinomial trees
- Know the difference between the binomial model for options on other types of assets (stock indices, stocks that pay dividends, bonds, foreign exchange, other derivatives)

Upper and lower bounds for options

Notation

S_0 = Current stock price
X = strike (exercise) price
T = Time to expiration of option
S_T = Stock price at maturity
r = risk free rate (continuously compounded)
C = Value of American call option
c = Value of European call option
P = Value of American put option
p = Value of European put option

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Upper and lower bounds

- Not dependent on any particular assumptions about the 6 factors that determine options prices (except r>0)

- If an option price is above the upper bound or below the lower bound, then there are profitable opportunities for arbitrageurs

Upper bounds - Call

- A European or American call gives the holder the right to buy one share of a stock for a certain price.
- No matter what happens, the option can never be worth more than the stock.
- Upper bound $c \leq S_0$ and $C \leq S_0$
- If this relation does not hold, an arbitrageur can make a riskless profit by buying the stock and selling the call option

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Upper bounds - Put

- A European or American put gives the holder a right to sell a stock for X
- No matter how low the stock price becomes, the option can never be worth more than X
- Upper bound $p \leq X$ and $P \leq X$

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Upper bounds- Put

- For European options, we know that at **maturity** the option cannot be worth more than X. This means that it cannot be worth more than the present value of X today

$$p \leq Xe^{-rT}$$

- If this does not hold, an arbitrageur could make a riskless profit by writing the option and investing the proceeds of the sale at the risk-free interest rate

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Lower bounds - call

- A lower bound for the price of a European call option on a non-dividend paying stock is

$$S_0 - Xe^{-rT}$$

- This can be shown by constructing 2 portfolios and examining the value of these at time 0 (today) and time T (maturity)

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Lower bounds - call

- Portfolio A: c (option) + Xe^{-rT} (cash)
- Portfolio B: 1 stock

	time 0	time T
A	$-c_0 - Xe^{-rT}$	$\max(S_T-X,0) + X$ $= \max(S_T,X)$
B	$-S_0$	S_T

Lower bounds - call

Since A≥B at time t, then A≥B must also be the case at t=0 (no arbitrage).

$$c_0 + Xe^{-rT} \geq S_0 \quad \Leftrightarrow \quad c_0 \geq S_0 - Xe^{-rT}$$

Since the worst case is that the option is worthless at maturity, the value can never be negative

$$c_0 \geq \max(S_0 - Xe^{-rT})$$

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Lower bounds - put

- The lower bound for a European put on a non-dividend paying stock is:

$$Xe^{-rT} - S_0$$

- This can be shown by constructing 2 portfolios and examining the value of these at time 0 (today) and time T (maturity)

Lower bounds - put

- Portfolio C: p (option) + 1 stock
- Portfolio D: Xe^{-rT} (cash)

	time 0	time T
C	$-p_0 - S_0$	$\max(X-S_T,0) + S_T$ $= \max(X,S_T)$
D	$- Xe^{-rT}$	X

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Lower bounds - put

Since A≥B at t=T, then A≥B must also be the case at t=0 (in the absence of arbitrage opportunities)

$$p_0 + S_0 \geq Xe^{-rT} \quad \Leftrightarrow \quad p_0 \geq Xe^{-rT} - S_0$$

Because the worst that can happen to a put option is that it expires worthless, its value cannot be negative

$$p_0 \geq \max(Xe^{-rT} - S_0)$$

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Put-call parity for options

Put-call parity

- Important relation between p and c
- Can be proven by examing portfolios A and C from the previous examples:
- Portfolio A: c (option) + Xe^{-rT} (cash)
- Portfolio C: p (option) + 1 stock

	time 0	time T
A	$-c_0 - Xe^{-rT}$	$\max(S_T, X)$
C	$-p_0 - S_0$	$\max(X, S_T)$

Put-call parity

- The portfolios have equal value at time T. Because they are European and can't be exercised before maturty, they must also have the same value at time 0

$$c_0 + Xe^{-rT} = p_0 + S_0$$

This relationship between c and p is called put-call parity. It says that the value of a european call with a certain exercise price and exercise dato can deduced from the value of a European put with the same strike price and maturity T, and vice versa.

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American options

- Put-call parity: For a non-dividend paying stock, it can be shown that:

- $S_0 - X \leq C - P \leq S_0 - Xe^{-rT}$

Early exercise

Early exercise - American call

- For an American call on a non-dividend paying stock it is *never* optimal to exercise before maturity

- Argument
  - If you intend to hold the stock to maturity it is better to hold the option
    - save money on the strike price (time value of money)
    - a certain probability that teh stock price falls below the strike price before maturity (insurance)
  - if you think that the stock is over-priced it is better to sell the option than to exercise it

Early exercise - American call

- Remember that

$c_0 \geq S_0 - Xe^{-rT}$

Since an American call has at least as many exercise opportunities as a European call then

$C_0 \geq c_0$

Since r>0, then

$C_0 > S_0 - Xe^{-rT}$

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Early exercise - American put

- It can be optimal to exercise an American put option on a non-dividend paying stock early. For an American put on a non-dividend paying stock it is *always* optimal to exercise before maturity as long as the option is sufficiently *in-the-money*

- Argument
  - If the strike price is 10 and the stock price is almost 0. If you exercise you would get approx. 10
  - By waiting until maturity you cannot get more than 10 (impossible). The profit may atually be less than 10.

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Early exercise - American put

- Remember that for a European put

$$p_0 \geq Xe^{-rT} - S_0$$

for an American put the condition is stronger

$$P_0 \geq X - S_0$$

because immediate exercise is possible

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Exercises (bounds, parity, early exercise)

1. What are the 6 factors that influence the price of an option?
2. What is the lower bound of a 4 month call on a stock when the stock price is 28, strike is 25 and the risk-free rate is 8% (pr year)?
3. What is the lower bound for a 1 month European put when the stock price is 12, strike is 15 and the risik-free rate is 6%?
4. Explain why early exercise of an American call on a non-dividend paying stock is not optimal?
5. Explain why early exercise of a European call on a non-dividend paying stock is not optimal?

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Factors affecting option prices

- There are six factors affecting the price of a stock option

1. The current stock price, S_0
2. The strike price, **X**
3. The time to expiration, **T**
4. The volatility of the stock price, σ
5. The risk free interest rate, **r**
6. The dividends expected during the life of the option, **q**

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Exercise 2

T = 4 months

type = call on a stock

S0 = 28

X = 25

r = 8% (pr year)

Lower bound:

$$c_0 \geq \max(S_0 - Xe^{-rT}, 0)$$
$$c_0 \geq \max(28 - 25e^{-0.08x4/12}, 0)$$
$$c_0 \geq \max(3.66, 0)$$
$$c_0 \geq 3.66$$

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Exercise 3

T = 1 month

Type = European put

S0 = 12

X = 15

r = 6%

Lower bound put:

$p_0 \geq \max(Xe^{-rT} - S_0, 0)$

$p_0 \geq \max(15e^{-(0.06x1/12} - 12, 0)$

$p_0 \geq \max(2.93, 0)$

$p_0 \geq 2.93$

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Early exercise - American call

- For an American call on a non-dividend paying stock it is *never* optimal to exercise before maturity

- Argument
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Binomial model

Binomial pricing model

- A simple and popular model for pricing options
- Building binomial trees
  - A diagram that shows the possible outcomes for a stock over the life time of an option
  - Assumes that the stock price follows random walk (i.e. random outcomes)
  - Over 1 time step the stock will either go up or down
  - Probabilities related to upward and downward move
    - Probability of upward movement of stock price (up-probability)
    - Probability of downward movement of stock price (down-probability)

Deriving the binomial model (2 approaches)

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Deriving the binomial model

- Approach 1 (Delta hedging): Portfolio of a shares and an option
  - The aim is to derive the binomial pricing formula by creating a portfolio of shares and an option in order to remove risk and thereby simplify the valuation

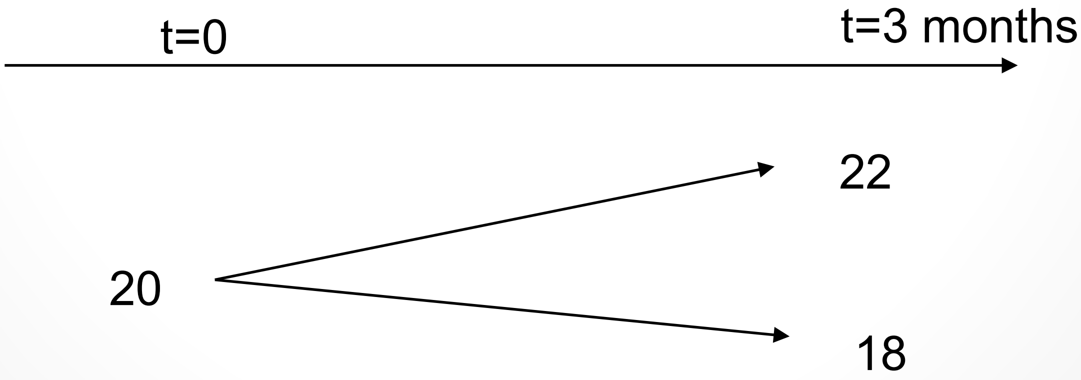
- Approach 2 (Replication): Replicating portfolios
  - The aim is to derive the binomial pricing formula by creating a portfolio of shares and bonds which mimics the cash flow from the option

Approach 1: Delta hedging

- Create a portfolio of x amount of shares and an option
- The amount x is chosen in order to eliminate uncertainty
- This simplifies the valuation

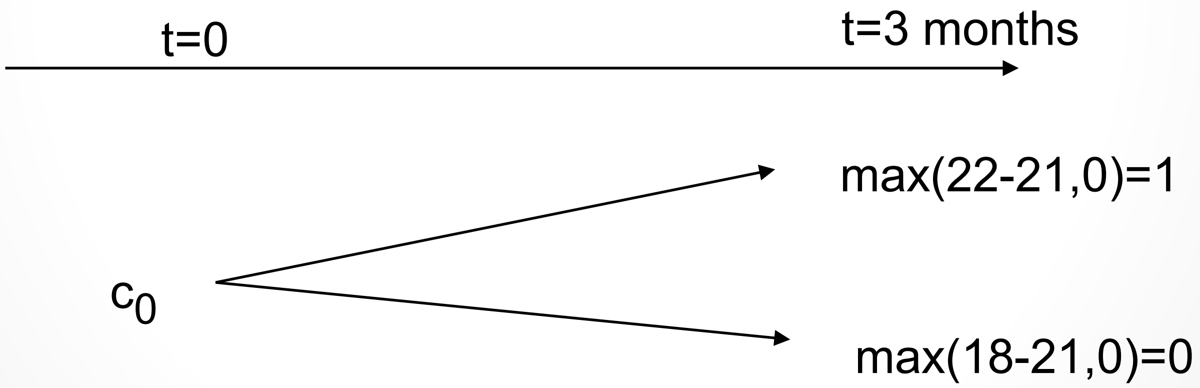
1-step model

- Today's stock price is 20
- It is known that in 3 months it will either be 18 or 22
- We want to price a European call option on the stock maturing in 3 months with a strike price of 21 (r=12%)



1-step model

- What is the value of the option in 3 months (at Maturity)?
- Value at maturity = c_T = max(S_T-X, 0)



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1-step model

- What is the value of the option today? c_0?
- It is the present value of c_T = max(S_T-X,0)
- How should we value the present value?
- The NPV of an expected cash flow with only 1 outcome:

$$NV_0 = \frac{CF_T}{(1+k)^T} \Leftrightarrow CF_T e^{-\mu T}$$

- The NPV of an expected cash flow with 2 possible outcomes, CF^1 og CF^2

$$NV_0 = \frac{p_1 CF_T^1 + p_2 CF_T^1}{(1+k)^T} \Leftrightarrow (p_1 CF_T^1 + p_2 CF_T^1) e^{-\mu T}$$

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1-step model

- What is the value of the option today? c_0?
- It is the present value of $c_T = \max(S_T - X, 0)$
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- The NPV of an expected cash flow with 2 possible outcomes, CF^1 og CF^2

$$NV_0 = \frac{p_1 CF_T^1 + p_2 CF_T^1}{(1+k)^T} \Leftrightarrow (p_1 CF_T^1 + p_2 CF_T^1)e^{-\mu T}$$

How do we calculate / estimate these?

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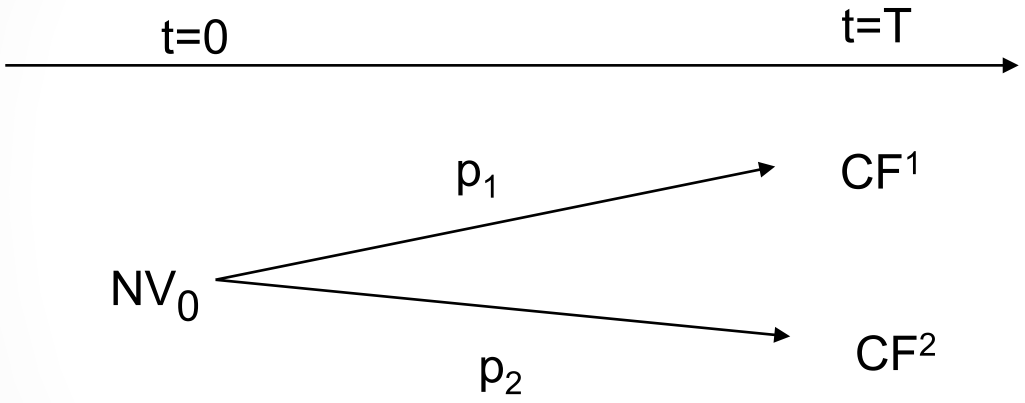
1-step model



What is p_1 og p_2 (the probabilities for an up-move or a down-move in the stock prices?
What is μ (cost of capital)? CAPM? WACC?

1-step model

- It can be shown that one can price options without having to calculate p and μ.
- We use the ’No-arbitrage’ argument and ‘risk neutral valuation’
- We construct a portfolio consisting of stocks and options (specific combination) such that there is no uncertainty around the value of the option in 3 months.
- We can therefore argue that since the portfolio has no risk (i.e. the outcome is know), we can discount the expected cash flow using the risk free interest rate.
- The cost of setting up the portfolio will therefore be equal to the price of the option

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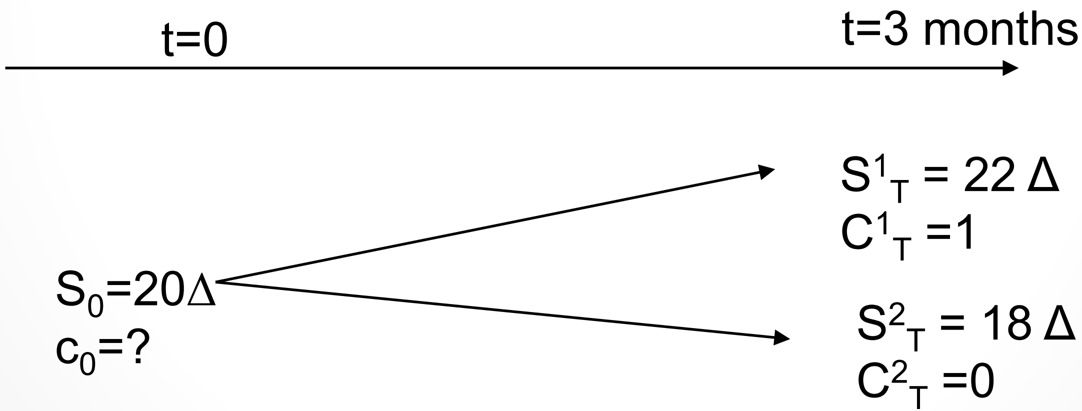
1-step model

- The portfolio consists of Δ stocks (long) and 1 call option (short)
- We have to calculate Δ such that the portfolio becomes riskless



1-step model

- Value of portfolio if stock increases: 22Δ – 1
- Value of portfolio if stock decreases: 18Δ – 0
- The portfolio is riskless if we select Δ such that the values of the portfolios in 3 months are identical if the stock goes up or down (i.e. no uncertainty in the outcome)
- $22\Delta - 1 = 18\Delta \Leftrightarrow \Delta = 1/4 = 0.25$
- The riskless portfolio consists of 0.25 stocks (long) and 1 option (short)

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1-step model

- Conclusion: Even if the stock price increases or decreases, the value of the portfolio is not affacted
-

Up-move: 22 => C^1_T= 22 x 0.25 – 1 = 4.5

Down-move: 18 => C^2_T= 18 x 0.25 -0 = 4.5

Riskless portfolios must, if there are no arbitrage opportunities, have a return equal to the risk free rate (cost of capital = risk free rate)

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1-step model

- The value of the portfolio today (V_0) there is:
  $V_0 = 4.5\ e^{-0.12x3/12} = 4.367$

- The value of the option today (c_0) will then be:
  $V_0 = \Delta S_0 - c_0$
  $c_0 = \Delta S_0 - V_0$
  $= 0.25 \times 20 - 4.367$
  $= 0.633$

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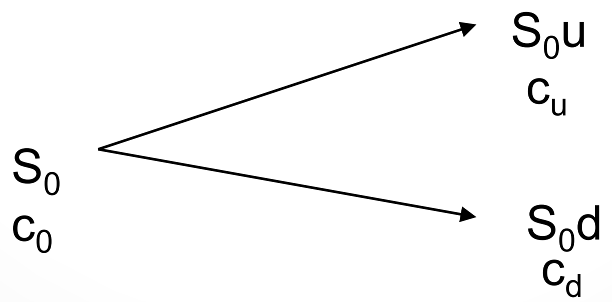
Mathematical derivation

- Notation:
  - u is up-factor (increase in stock price): u > 1 (u-1 => % increase)
  - d is down-factor (decrease in stock price): d < 1
  - S_0u = stock price after up-move
  - S_0d = stock price after down-move
  - c_u is the value of the option after a up-move
  - c_d is the value of the option after a down-move



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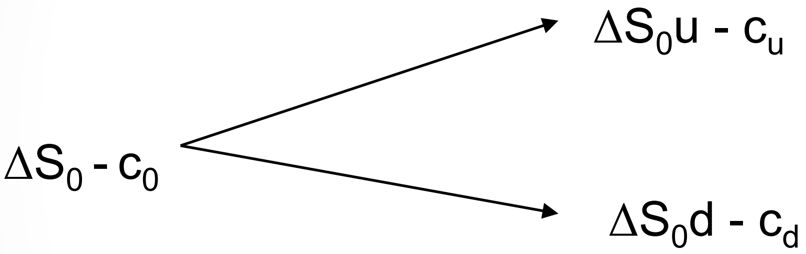
Mathematical derivation

- A portfolio of Δ stocks (long) and 1 call option (short)



- Δ is set such that the portfolio is risk free

$$\Delta S_0 u - c_u = \Delta S_0 d - c_d$$

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Mathematical derivation

$\Delta S_0 u - c_u = \Delta S_0 d - c_d$

Solve with respect to (wrt) to Δ:

$$\Delta = \frac{c_u - c_d}{S_0 u - S_0 d}$$

Since the portfolio is risk free we can find its value today (present value):

$$V_0 = (\Delta S_0 u - c_u) e^{-rT}$$

The cost of setting up the portfolio (equal to V_0):

$$\Delta S_0 - c_0$$

Mathematical derivation

- Since the portfolio is risk free, the present value of the portfolio is equal to the cost of constructing the portfolio (discounted with the risk free rate)

$$V_0 = (\Delta S_0 u - c_u)e^{-rT} = \Delta S_0 - c_0$$

- Solve wrt c_0 gives

$$c_0 = \Delta S_0 (1 - ue^{-rT}) + c_u e^{-rT}$$

- Replace Δ in above equation with this

$$\Delta = \frac{c_u - c_d}{S_0 u - S_0 d}$$

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Mathematical derivation

- by simplifying we get:

$$c_0 = e^{-rT}\left[q \times c_u + (1-q) \times c_d\right]$$

- where q represents:

$$q = \frac{e^{rT} - d}{u - d}$$

- where c_u and c_d represent:

c_u = max (S_0u - X, 0)
c_d = max (S_0d - X, 0)

What does this mean?

$$c_0 = e^{-rT}\left[q \times c_u + (1-q) \times c_d\right]$$

Price of a call option

discounting

'probability' of up-move

value(payoff) of option if stock price increases

'probability' of down-move

value(payoff) of option is stock price decreases

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Do you remember this?

$$NV_0 = \frac{p_1 CF_T^1 + p_2 CF_T^1}{(1+k)^T} \Leftrightarrow (p_1 CF_T^1 + p_2 CF_T^1)e^{-\mu T}$$

- In this equation we lacked both p1 and p2, and μ, the discount rate
- Now we have found them...........Or not?

But.....

- When we made the portfolio risk free we could discount the payoffs (cash flows) from the option using the risk free discount rate
- BUT! It is important to realize that the probabilities *p1* and *p2* are not equal to *q1* and *q2* (*q2*=1-*q1*).
- However, q1 and q2 are interpreted as probabilities (but are really just simplifications of the formula)

- p1 & p2 => ***actual*** up- and down- probabilities (real)
- q1 & q2 => ***risk neutral*** up- and down- probabilities (interpreted)

Risk neutral valuation

- In a risk free world all individuals are indifferent to risk
- Investors require no compensation for risk
- The expected return on all assets is the risk free rate
- Risk neutral valuation: we can assume that the world is risk neutral when pricing options
- This may seem a bit strange and unrealistic, but it is important to realize that the prices we calculate using risk neutral valuation are **correct** both in <u>a risk neutral</u> and in <u>the real world</u>

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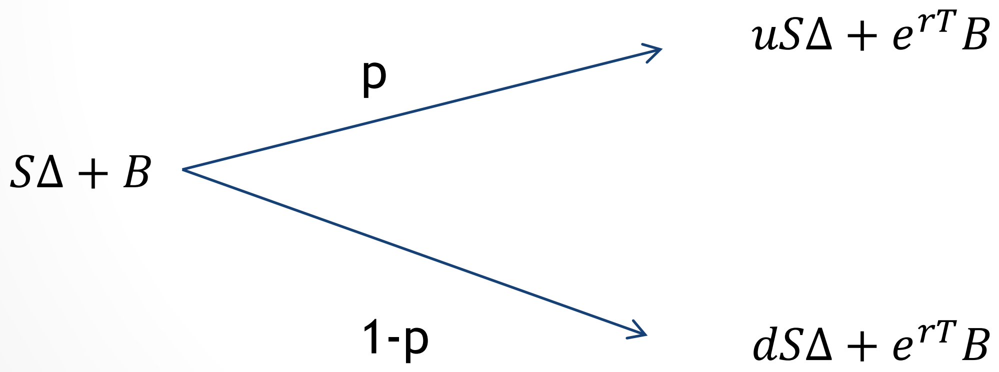
Approach 2: Replicating portfolios

- Buy a number of shares, Δ, and invest B in bonds
- Outlay for portfolio today is SΔ + B
- The tree shows the possible values one period later



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Replicating portfolios

- Choose Δ, B so that the portfolio replicates the call option
- By replicate we mean duplicate or mimic the behaviour of the option (cash flows)
- We get two equations

$$uS\Delta + e^{rT}B = c_u$$
$$dS\Delta + e^{rT}B = c_d$$

- The solutions are

$$\Delta = \frac{c_u - c_d}{(u-d)S} \qquad \mathrm{B} = \frac{uc_u - dc_d}{(u-d)e^{rT}}$$

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Replicating portfolios

- (Δ,B) gives the same values in both up and down states
- They must therefore have the same value now

$$c = S\Delta + B$$

$$= \frac{(c_u - c_d)e^{rT} + uc_u + dc_d}{(u-d)e^{rT}}$$

$$= \frac{(e^{rT} - d)c_u + (u - e^{rT})c_d}{(u-d)e^{rT}}$$

Replicating portfolios

- Define

$$q \equiv \frac{(e^{rT} - d)}{u - d}$$

- Rewrite the formula as

$$c_0 = e^{-rT}\left[q \times c_u + (1 - q) \times c_d\right]$$

- Which is the same as using Approach 1 (Hull)

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Example

- Value a 3 month call option on a non-dividend paying stock. The current stock price is 20. The strike price is 21. The risk free rate is 12%. In 3 months the price will either be 18 or 22.

- $T = 3/12$
- $S_0 = 20$
- $X = 21$
- $r = 12\%$
- $c_0 = ?$
- $u = 22/20 = 1.1$
- $d = 18/20 = 0.9$

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Example

- Value a 3 month call option on a non-dividend paying stock. The current stock price is 20. The strike price is 21. The risk free rate is 12%

- Risk neutral probabilities

$$q = \frac{e^{rT} - d}{u - d}$$

- Call option price

$$c_0 = e^{-rT}\left[q \times c_u + (1-q) \times c_d\right]$$

Example

- Value a 3 month call option on a non-dividend paying stock. The current stock price is 20. The strike price is 21. The risk free rate is 12%

- Risk neutral probabilities

$$q = \frac{e^{0.12x3/12} - 0.9}{1.1 - 0.9} = 0.652$$

- Call option price

$$c_0 = e^{-rT}\left[q \times c_u + (1-q) \times c_d\right]$$

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Example

- Value a 3 month call option on a non-dividend paying stock. The current stock price is 20. The strike price is 21. The risk free rate is 12%

  - Risk neutral probabilities

$$q = \frac{e^{0.12x3/12} - 0.9}{1.1 - 0.9} = 0.652$$

- Call option price

$$c_0 = e^{-0.12x3/12}\begin{bmatrix} 0.652 \times \max(22-21,0) \\ +(1-0.652) \times \max(18-21,0) \end{bmatrix}$$

Example

- Value a 3 month call option on a non-dividend paying stock. The current stock price is 20. The strike price is 21. The risk free rate is 12%

- Risk neutral probabilities

$$q = \frac{e^{0.12x3/12} - 0.9}{1.1 - 0.9} = 0.652$$

- Call option price

$$c_0 = e^{-0.12x3/12}\left[0.652 \times 1 + (1 - 0.652) \times 0\right]$$
$$= 0.633$$

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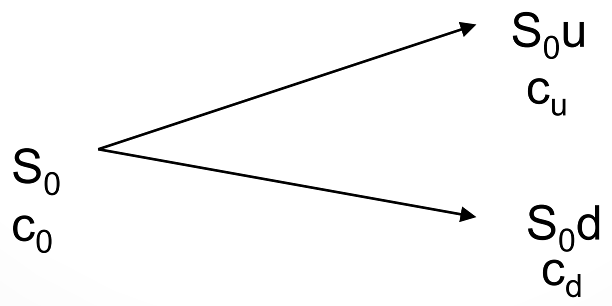
Applying the 1-step binomial tree

Binomial pricing model

- A simple and popular model for pricing options
- Building binomial trees
  - A diagram that shows the possible outcomes for a stock over the life time of an option
  - Assumes that the stock price follows random walk (i.e. random outcomes)
  - Over 1 time step the stock will either go up or down
  - Probabilities related to upward and downward move
    - Probability of upward movement of stock price (up-probability)
    - Probability of downward movement of stock price (down-probability)

Mathematical derivation

- Notation:
  - u is up-factor (increase in stock price): $u > 1$ (u-1 => % increase)
  - d is down-factor (decrease in stock price): $d < 1$
  - S_0u = stock price after up-move
  - S_0d = stock price after down-move
  - c_u is the value of the option after a up-move
  - c_d is the value of the option after a down-move



Mathematical derivation

- by simplifying we get:

$$c_0 = e^{-rT}\left[q \times c_u + (1-q) \times c_d\right]$$

- where q represents:

$$q = \frac{e^{rT} - d}{u - d}$$

- where c_u and c_d represent:

c_u = max (S_0u - X, 0)
c_d = max (S_0d - X, 0)

What does this mean?

$$c_0 = e^{-rT}\left[q \times c_u + (1-q) \times c_d\right]$$

Price of a call option

discounting

'probability' of up-move

value(payoff) of option if stock price increases

'probability' of down-move

value(payoff) of option is stock price decreases

Risk neutral valuation

- In a risk free world all individuals are indifferent to risk
- Investors require no compensation for risk
- The expected return on all assets is the risk free rate
- Risk neutral valuation: we can assume that the world is risk neutral when pricing options
- This may seem a bit strange and unrealistic, but it is important to realize that the prices we calculate using risk neutral valuation are **correct** both in <u>a risk neutral</u> and in <u>the real world</u>

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Two approaches for deriving the binomial price model

- «Delta hedging approach»
  - Remove uncertainty through delta hedging (delta hedging = choosing the number of stocks in order to eliminate risk)
  - Simplifies valuation (no need to calculate «real» probabilities and no need for risk adjustment of the discount rate (discount rate = risk free rate)
  - This is also an approach that is used to derive the Black-Scholes-Merton model

- «Replicating portfolio approach»
  - Choose a portfolio of stocks and bonds in order to mimic cash flow

Option pricing: methods

- Method 1: Analytical solution (pricing equation, closed form)
  - Black-Scholes model (1973): Options on stocks that do not pay dividends
  - Merton (1973): Options on stocks paying a known dividend or yield
  - Variants of BSM model:
    - Currency options (Garman and Kohlhagen, 1983), bonds, assets that pays a yield
    - Options on futures: Black'76 (1976)
  - Margrabe (1978): options on price spreads (no strike price)

- Method 2: Approximations
  - Kirk (1995): Options on price spreads (with strike price)
  - Bjerksund and Stensland (2002): American options

For more pricing formules see Haug: The Complete Guide to Option Pricing Formulas

Option pricing: methods

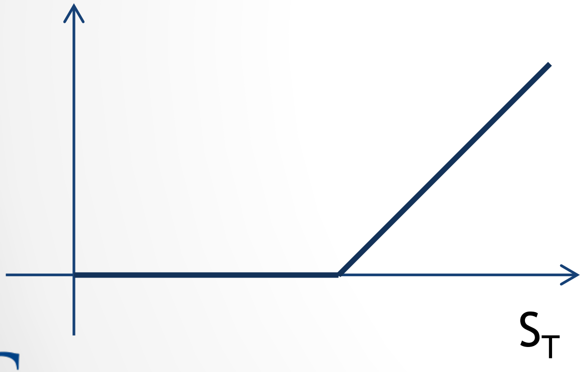
- Method 3: Numerical solutions
  - more flexible than analytical solutions

- Trees
  - Binomial trees (Cox-Ross-Rubinstein, 1979)
  - Trinomial trees (Boyle, 1986)

- Monte Carlo simulation
  - Find price process (mathematical representation of price behaviour)
  - Operationalise the price process
  - Find parameters for your model
  - Simulation of price paths
  - Valuation using payoff function

General idea

payoff



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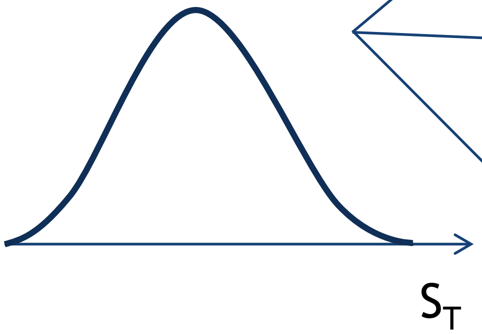
Need to model price behaviour and uncertainty



Closed form (analytical) solutions are derived mathematically

Discretized using time-steps (binomial and trinomial model)

Simulation (Monte Carlo)

$$\max(S_T - X, 0)$$ → **Valuation**

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2-step Binomial tree

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2-step model

- Today's stock price is 20
- In 3 months it is either 22 or 18 (1 time step)
- In 6 months it is either 24.2, 19.8 or 16.2
- The risk free rate is 12%
- The strike is 21
- What is the price of a European call with maturity 6 months?

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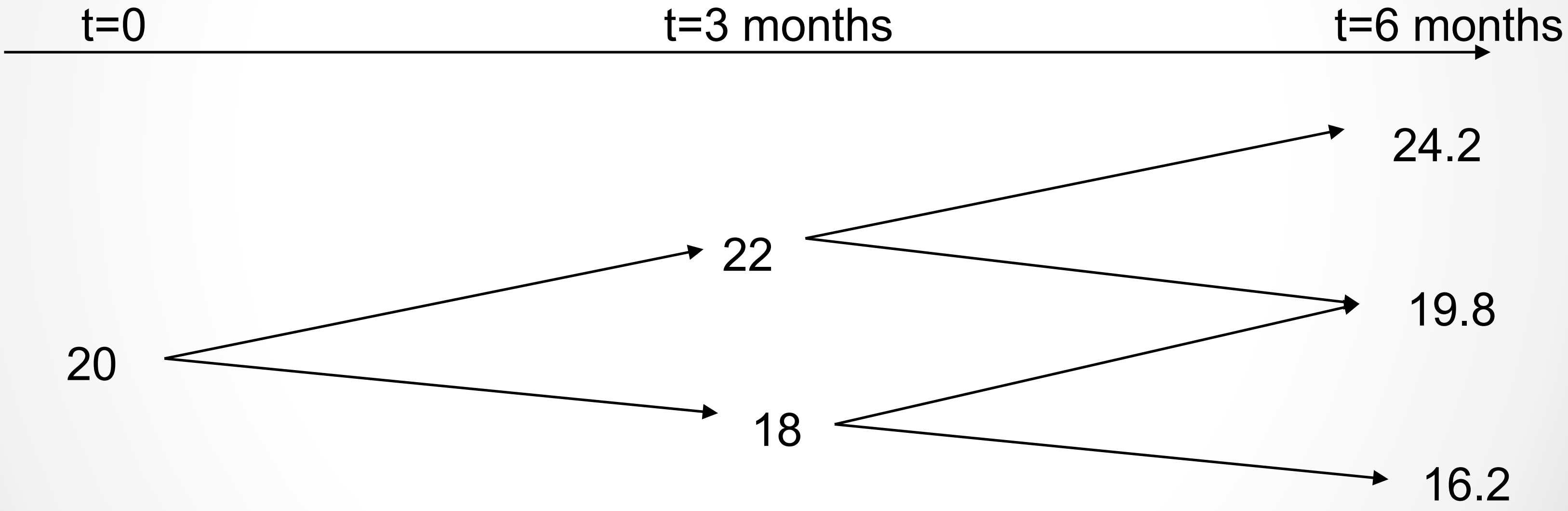
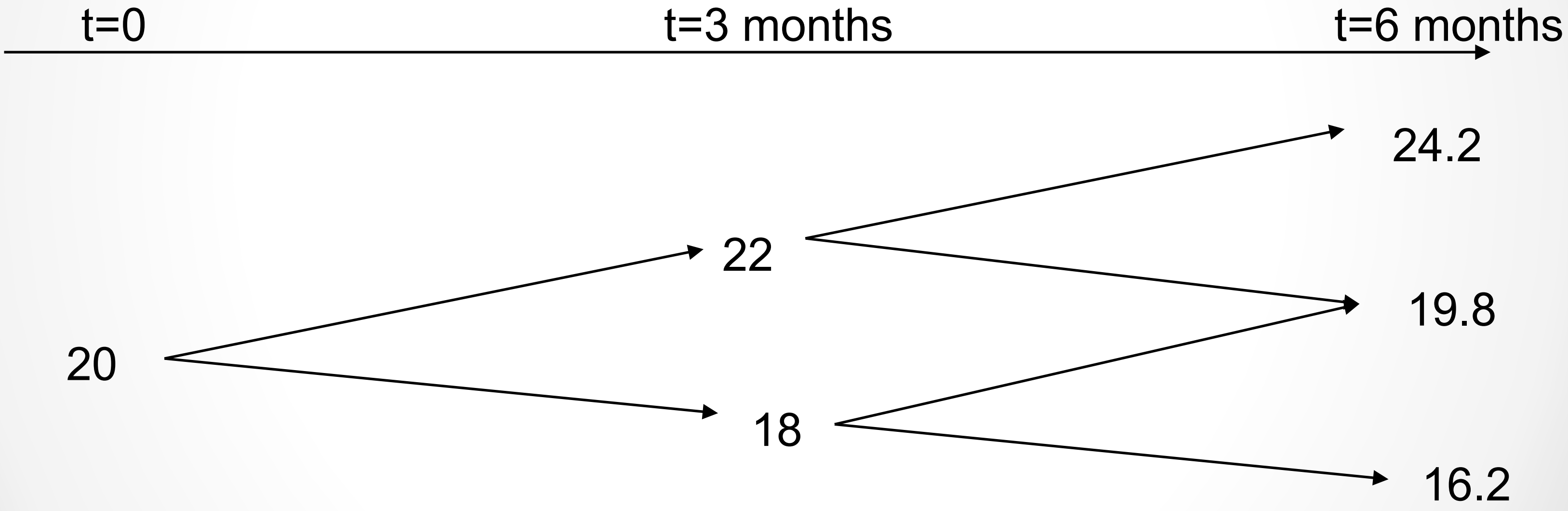
2-step model



- How is the expected spot price movement?

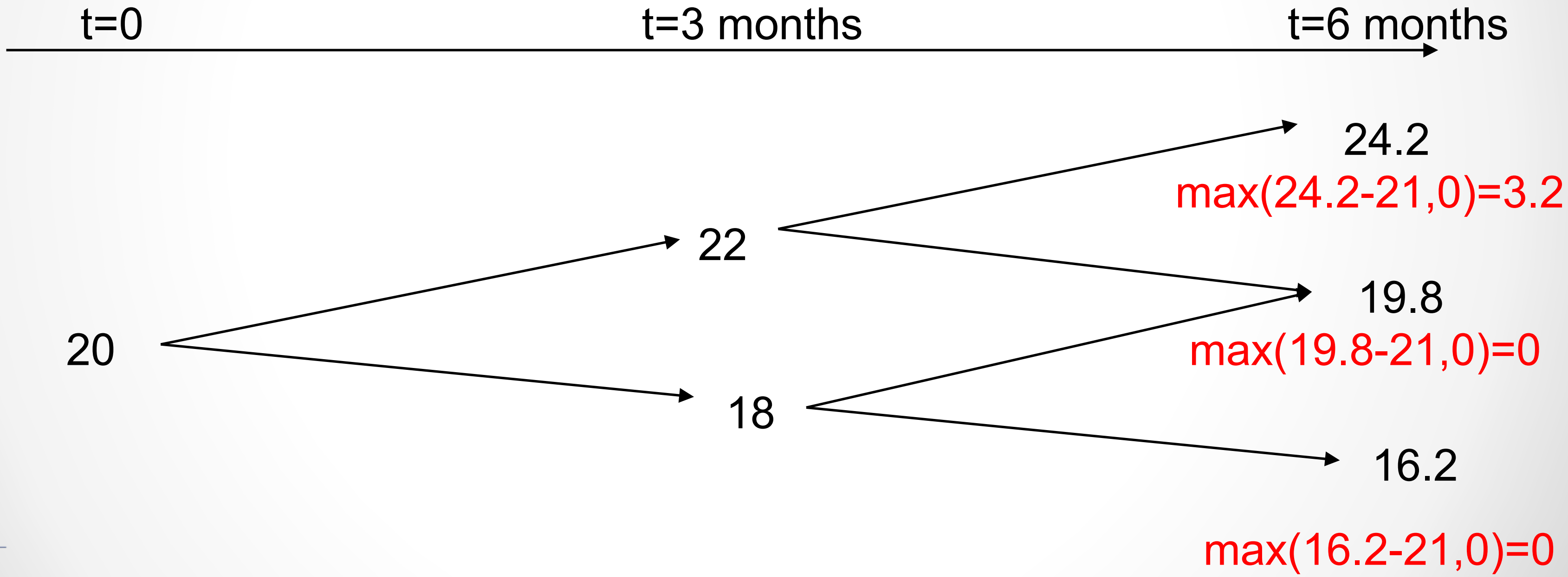
2-step model



- What is the price of the option? Start at Maturity, roll back to t=0

2-step model

Value of call at maturity: max (S_T-X, 0)



2-step model

Value of call at t = 3 months:

$$c_0 = e^{-rT}\left[q \times c_u + (1-q) \times c_d\right] \qquad q = \frac{e^{rT} - d}{u - d}$$

- Step 1: calculate the risk neutral probabilities:

$$q = \frac{e^{0.12x(3/12)} - 0.9}{1.1 - 0.9} = 0.6523$$

1-q = 1- 0.6523 = 0.3477

- Then calculate the value of the option at t=3 months (both nodes):

$S_{t=0.25}$ = 22: $$c_{t=0.25} = e^{-0.12x(3/12)}\left[0.6523 \times 3.2 + 0.3477 \times 0\right] = 2.0257$$

$S_{t=0.25}$ = 18: $$c_{t=0.25} = e^{-0.12x(3/12)}\left[0.6523 \times 0 + 0.3477 \times 0\right] = 0$$

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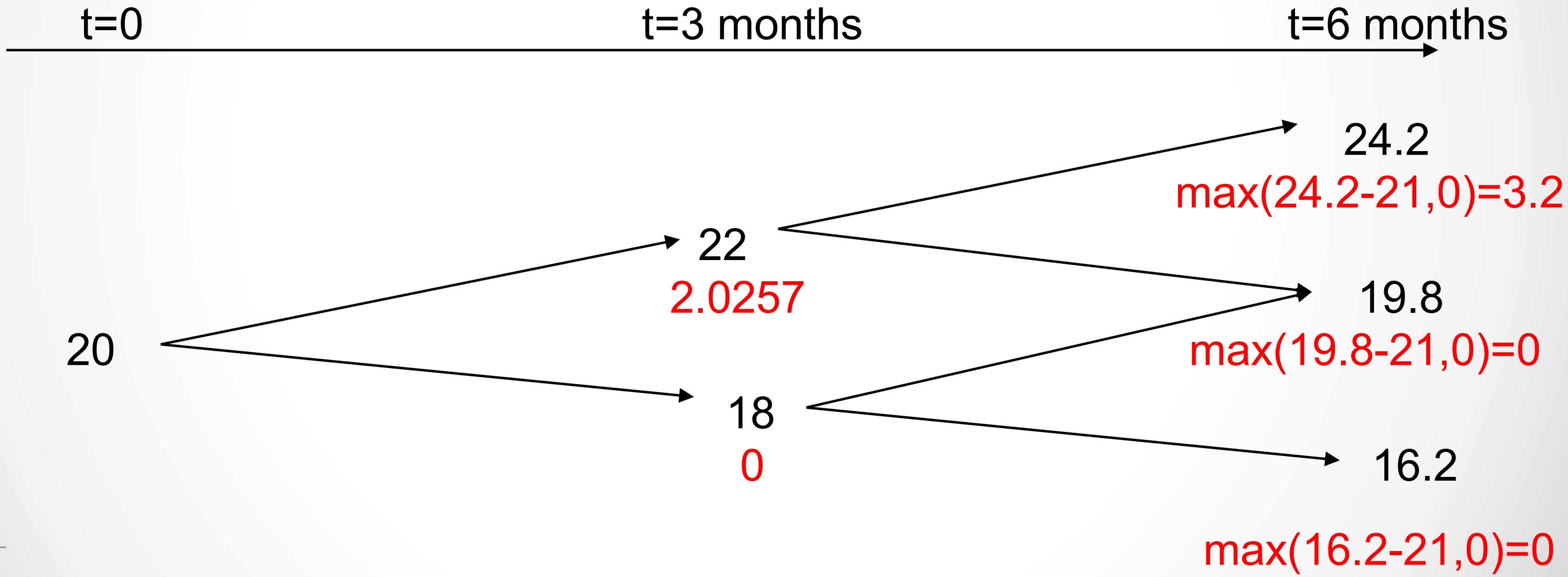
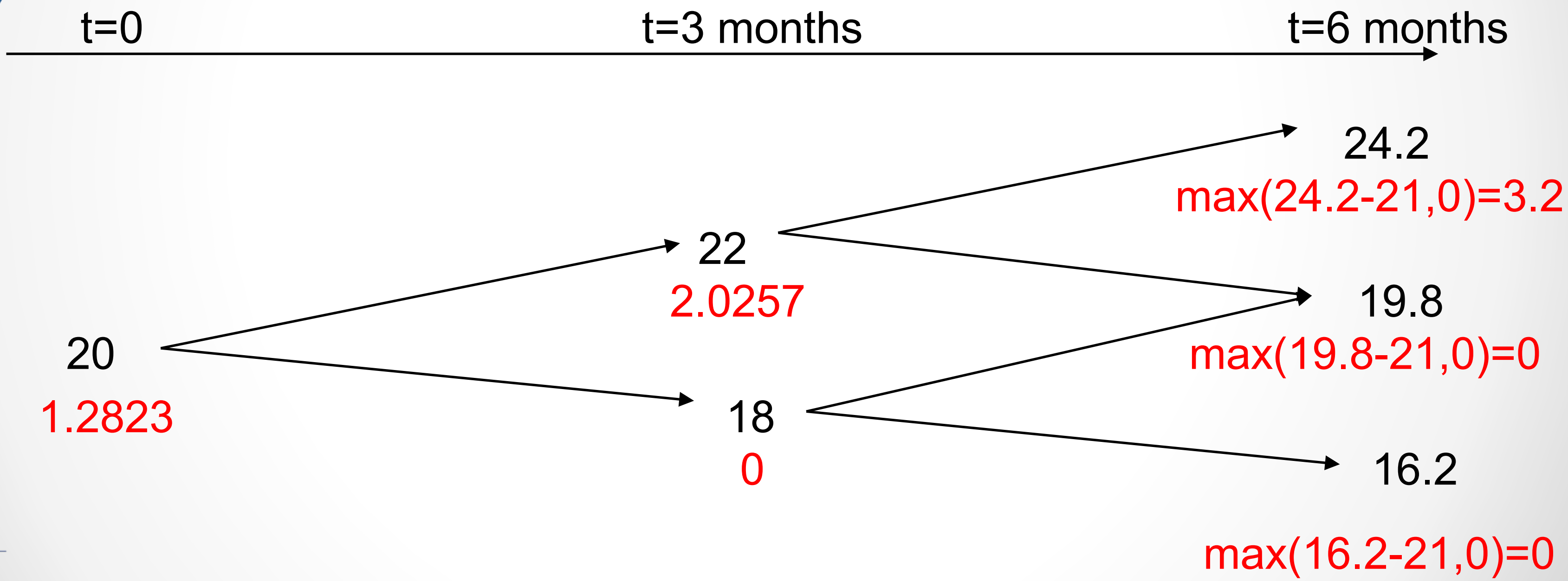
2-step model

Value of call at t = 3 months:



2-step model

Value of call at t = 0:



$$c_0 = e^{-0.12x(3/12)}[0.6523 \times 2.0257 + 0.3477 \times 0] = 1.2823$$

n-step Binomial tree

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Generalisation

- Definition:
  - nodes, start node & end noder
  - price path
- Generalised equations

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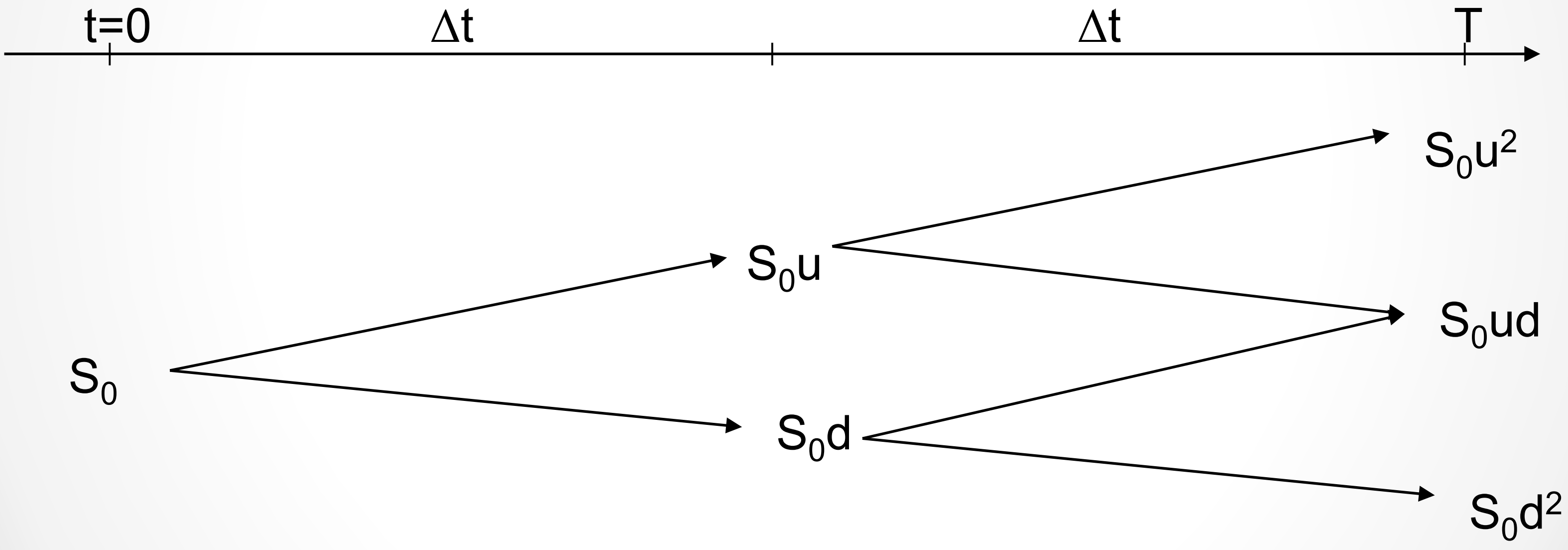
Definitions

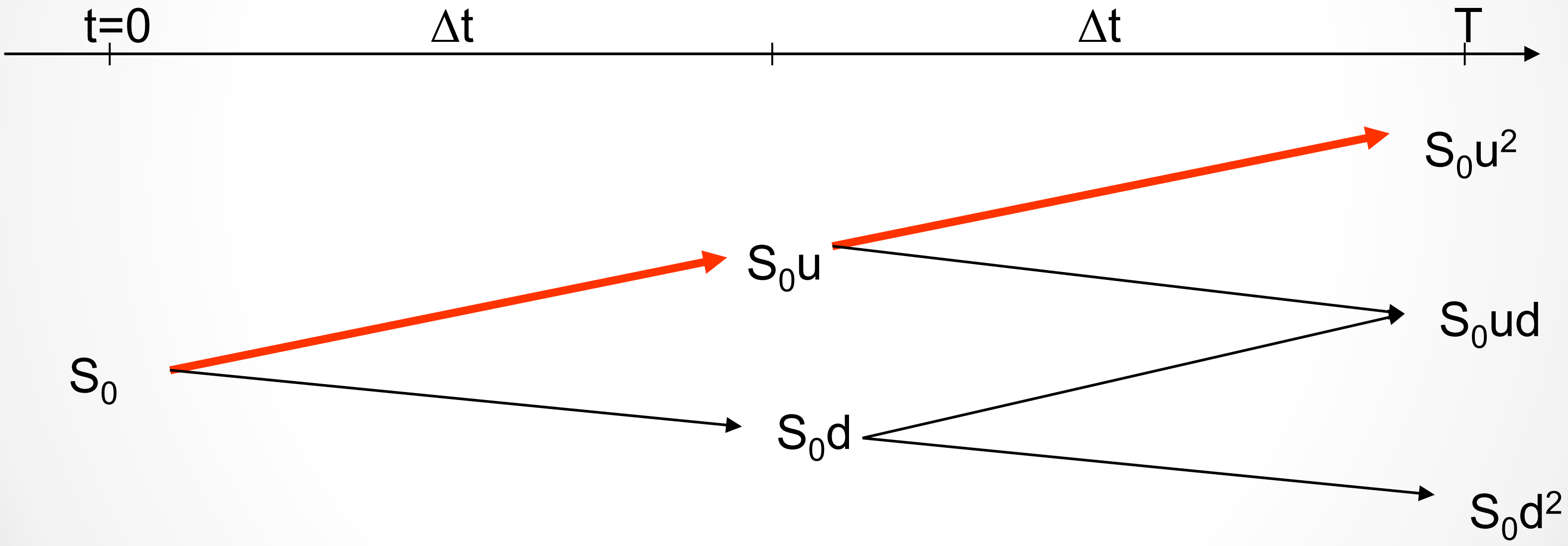
- Possible price paths



4 possible price paths

Definitions

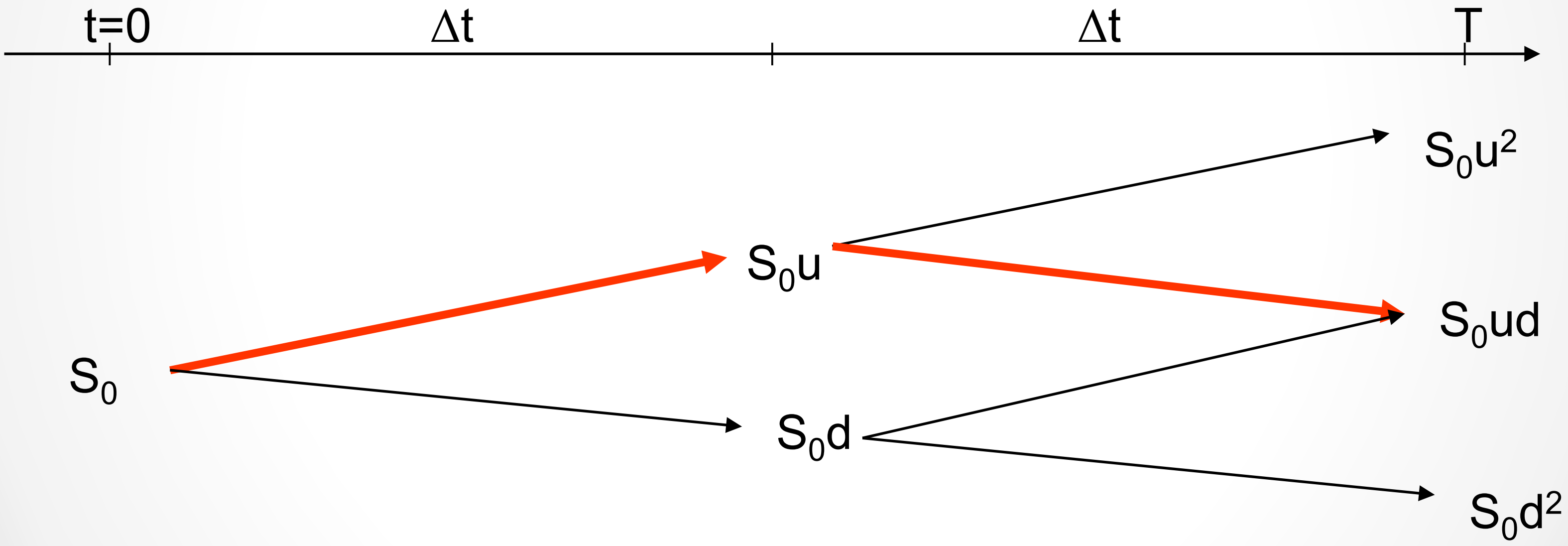
- Possible price paths



4 possible price paths

Definitions

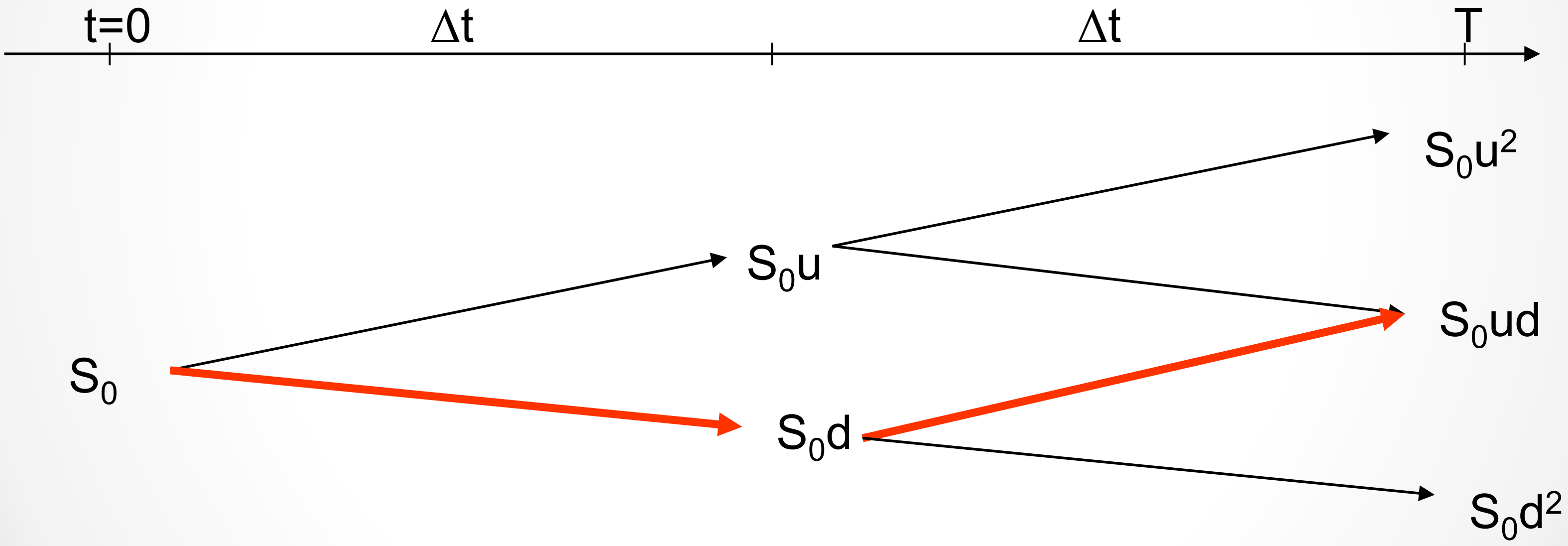
- Possible price paths



4 possible price paths

Definitions

- Possible price paths



4 possible price paths

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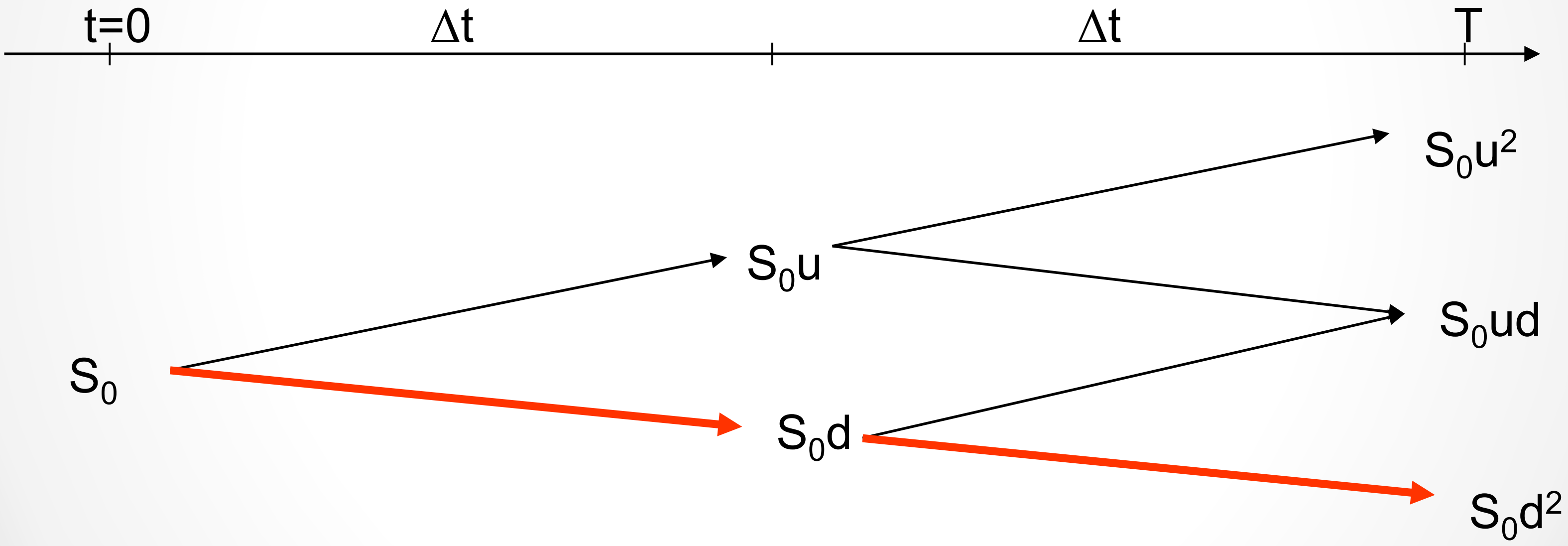
Definitions

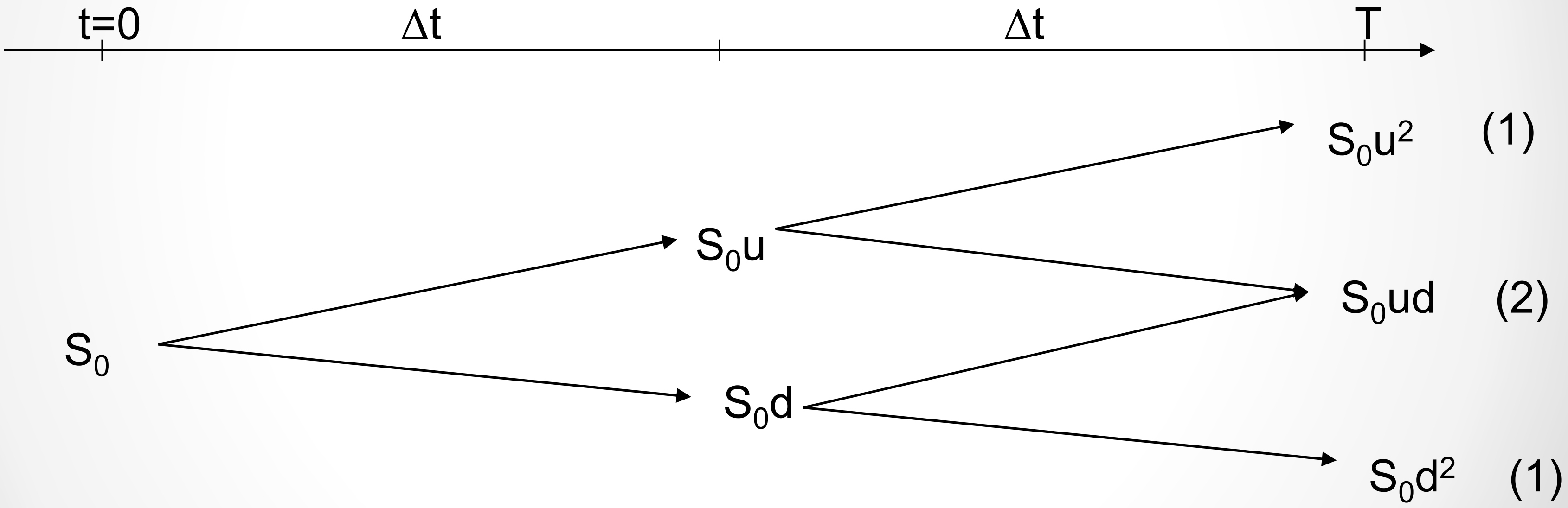
- Possible price paths



4 possible price paths

Definitions

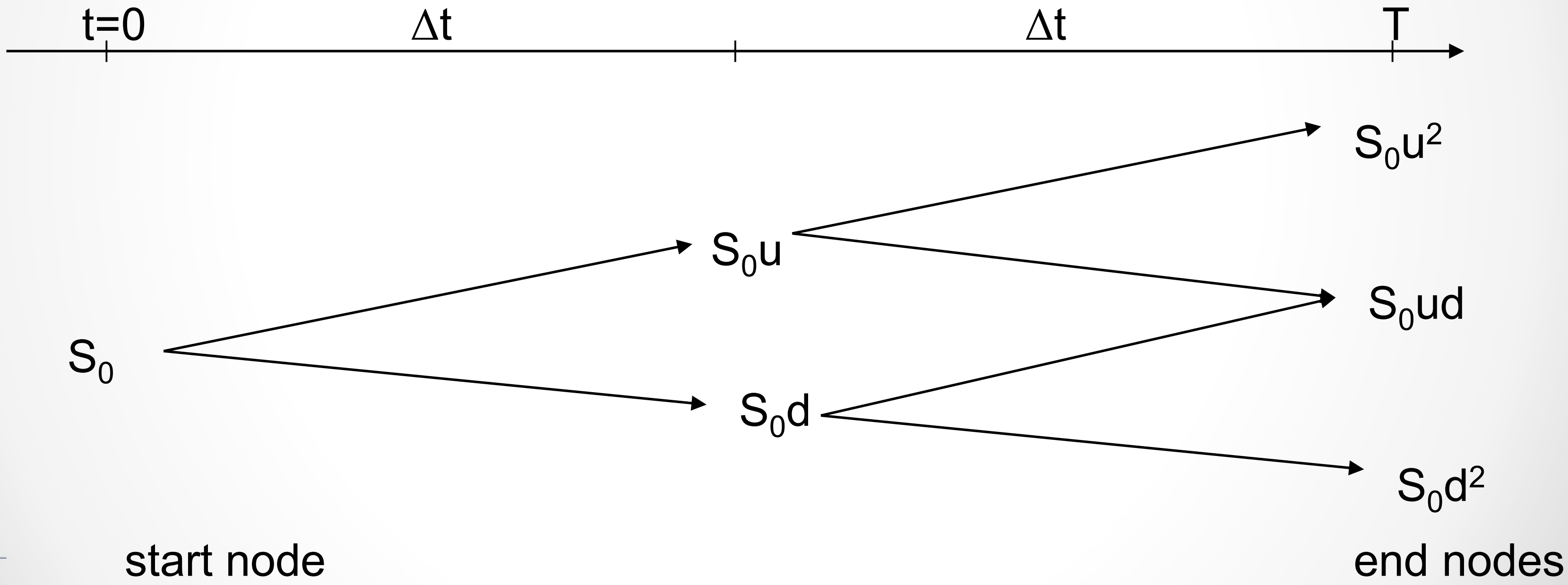
- Possible price paths



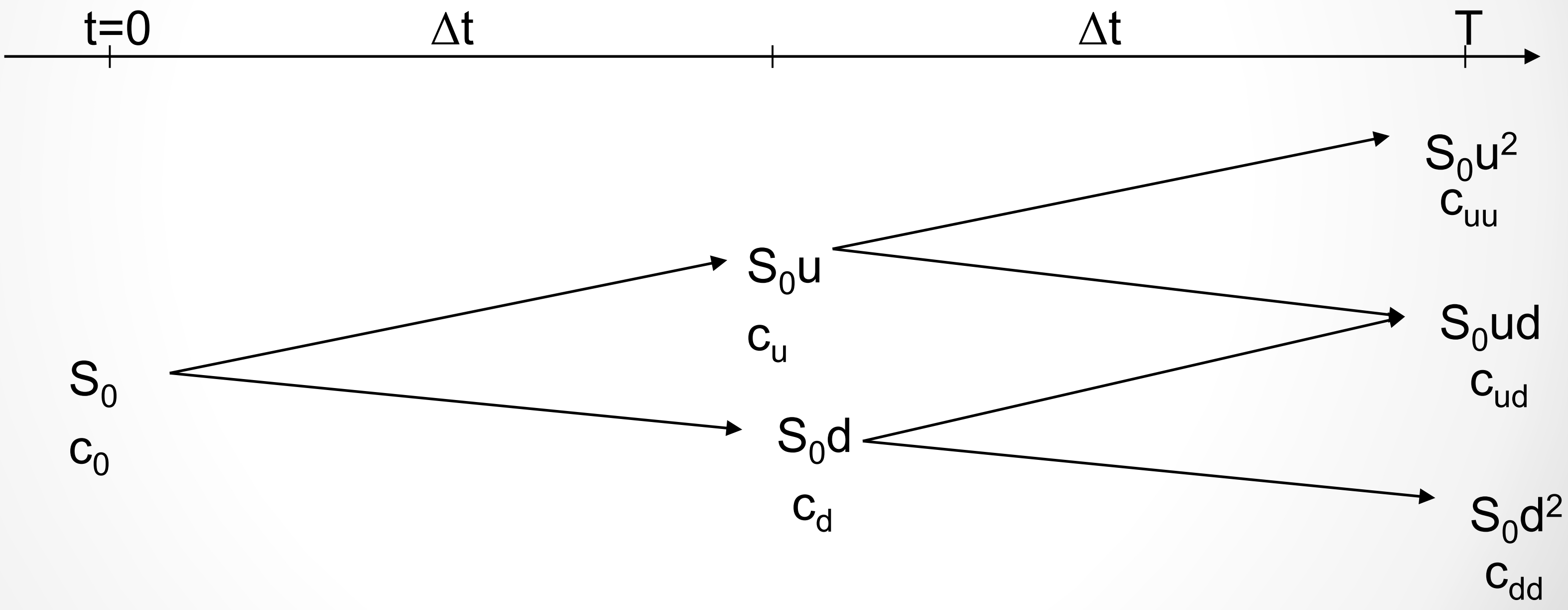
4 possible price paths

Definitions

- Nodes



Generalisation



Generalisation

- We set the lenght of the time step to Δt. The value of the option today is then:

$$c_0 = e^{-r\Delta t}\left[qc_u + (1-q)c_d\right]$$

$$q = \frac{e^{r\Delta t} - d}{u - d}$$

- The values of the option after 1 time step are:

$$c_u = e^{-r\Delta t}\left[qc_{uu} + (1-q)c_{ud}\right]$$

$$c_d = e^{-r\Delta t}\left[qc_{ud} + (1-q)c_{dd}\right]$$

Generalisation

- Replacing c_u and c_d in

$$c_0 = e^{-r\Delta t}\left[qc_u + (1-q)c_d\right]$$

- we arrive at

$$c_0 = e^{-2r\Delta t}\left[q^2 c_{uu} + 2q(1-q)c_{ud+}(1-q)^2 c_{dd}\right]$$

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Generalisation

- Even more general, the value of a European option can be calculated as:
  - The value of a European call (n-step):

$$c_t = e^{-r(T-t)} \sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} \times c_{n,i}$$

- where,

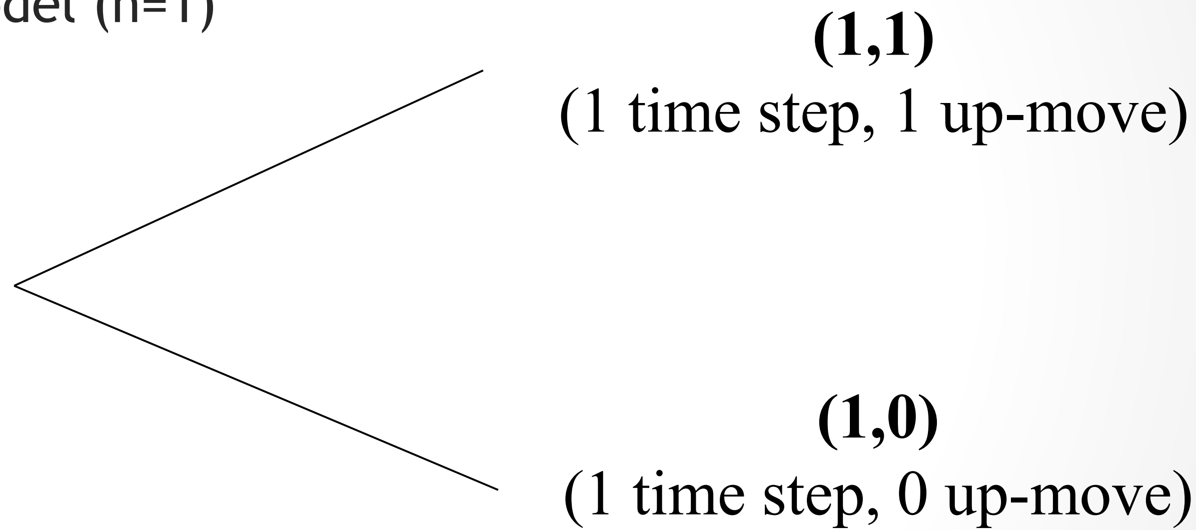
i = number of up-moves

n = number of time steps

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Example

- 1-step model (n=1)



Example

- 1-step model

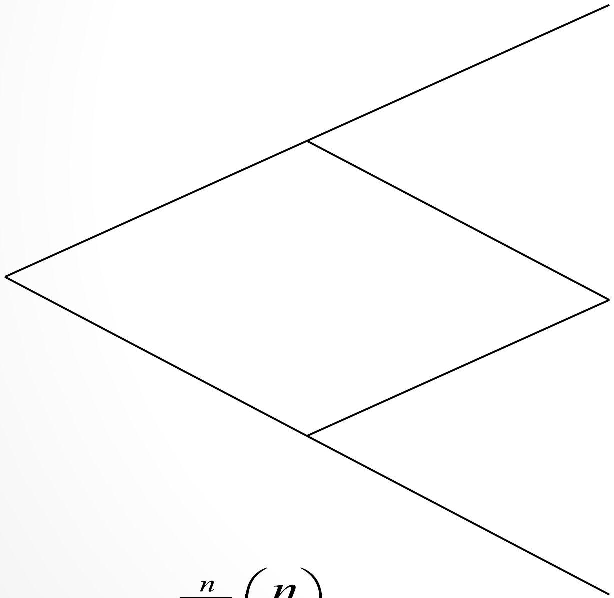
$$\binom{n}{i} = \binom{1}{1} = \frac{1!}{1!\,(1-1)!} = 1$$

$$c_t = e^{-r(T-t)} \sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} \times c_{n,i}$$

$$\binom{n}{i} = \binom{1}{0} = \frac{1!}{0!\,(1-0)!} = 1$$

Example

- 2-step model



$$\binom{n}{i} = \binom{2}{2} = \frac{2!}{2!(2\text{-}2)!} = 1$$

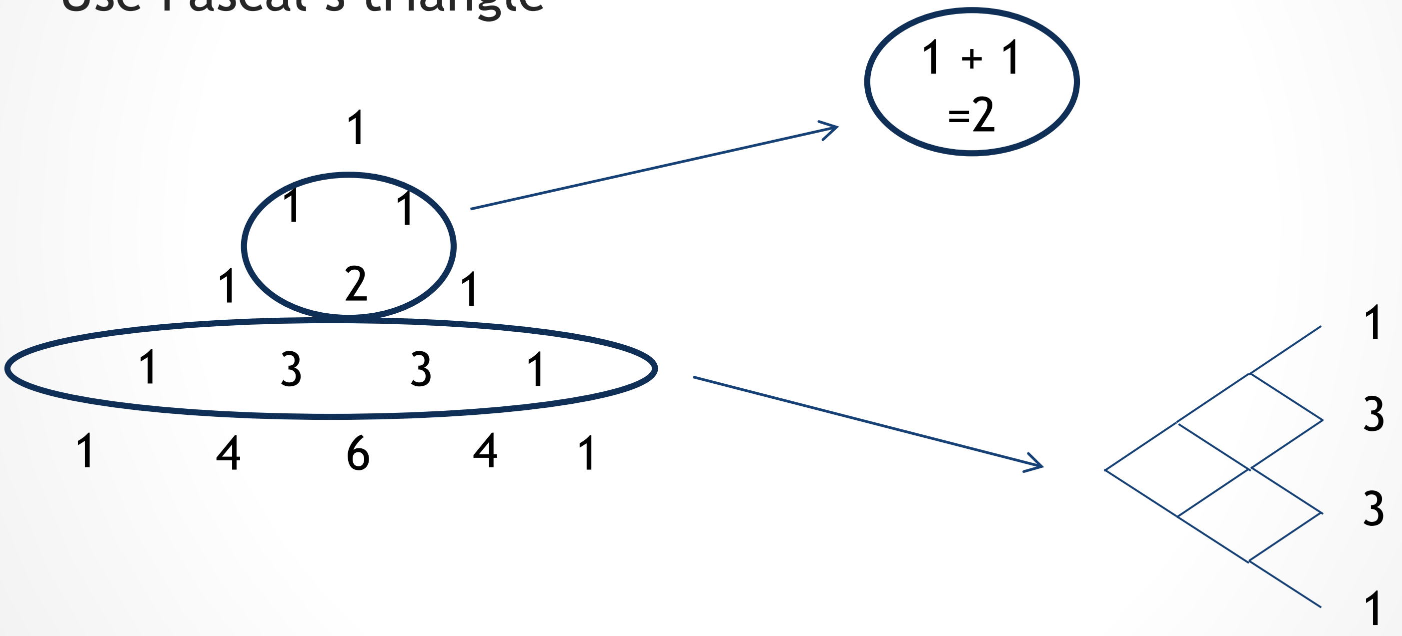
$$\binom{n}{i} = \binom{2}{1} = \frac{2!}{1!(2\text{-}1)!} = 2$$

$$\binom{n}{i} = \binom{2}{0} = \frac{2!}{0!(2\text{-}0)!} = 1$$

$$c_t = e^{-r(T-t)} \sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} \times c_{n,i}$$

Approach 2

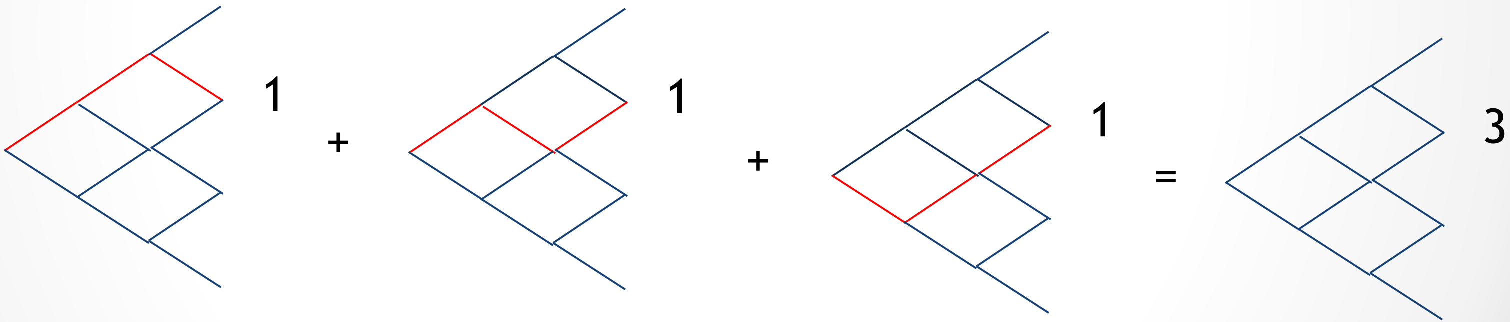
- Use Pascal's triangle



3-step tree

Approach 3

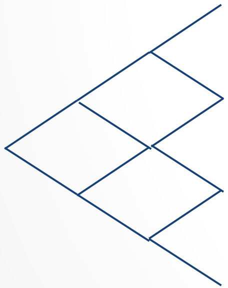
- Count the number of price paths



End result

$$c_t = e^{-r(T-t)} \sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} \times c_{n,i}$$

3-step tree



1 → $1 \times q^3(1-q)^{3-3} \times \max(S_{u^3 d^0} - X, 0)$

3 → $3 \times q^2(1-q)^{3-2} \times \max(S_{u^2 d^1} - X, 0)$

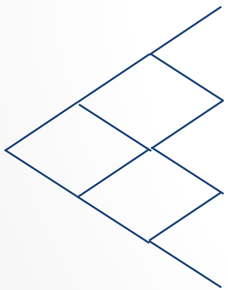
3 → $3 \times q^1(1-q)^{3-1} \times \max(S_{u^1 d^2} - X, 0)$

1 → $1 \times q^0(1-q)^{3-0} \times \max(S_{u^0 d^3} - X, 0)$

End result

$$c_t = e^{-r(T-t)} \sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} \times c_{n,i}$$

3-step tree



1 → $1 \times q^3 \times \max(S_{u^3} - X, 0)$

3 → $3 \times q^2 (1-q)^{3-2} \times \max(S_{u^2 d} - X, 0)$

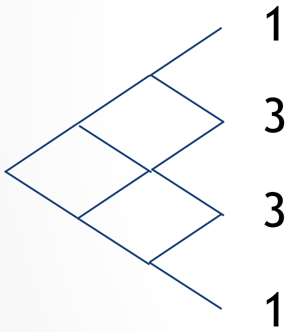
3 → $3 \times q^1 (1-q)^{3-1} \times \max(S_{u^1 d^2} - X, 0)$

1 → $1 \times (1-q)^3 \times \max(S_{d^3} - X, 0)$

Interpetation

$$c_t = e^{-r(T-t)} \sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} \times c_{n,i}$$

3-step tree



$$3 \times q^2 (1-q)^{3-2} \times \max(S_{u^2 d} - X, 0)$$

- 3: Number of price paths possible to reach end node
- $q^2 (1-q)^{3-2}$: ‘probability’ of arriving at that node for each of the price paths
- $\max(S_{u^2 d} - X, 0)$: Payoff (cash flow) from end node

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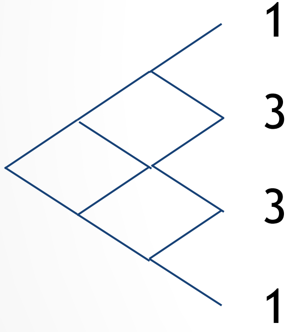
Interpetation

$$c_t = e^{-r(T-t)} \sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} \times c_{n,i}$$

3-step tree



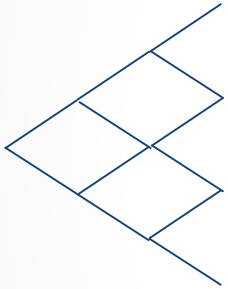
$$3 \times q^2 (1-q)^{3-2} \times \max(S_{u^2 d} - X, 0)$$

Total ‘probability’ of arriving at that node for each of the price paths

Payoff (cash flow) from end node

Interpretation

3-step tree



$$1 \rightarrow 1 \times q^3 \times \max(S_{u^3} - X, 0)$$

$$3 \rightarrow 3 \times q^2(1-q)^{3-2} \times \max(S_{u^2 d} - X, 0)$$

$$3 \rightarrow 3 \times q^1(1-q)^{3-1} \times \max(S_{u^1 d^2} - X, 0)$$

$$1 \rightarrow 1 \times (1-q)^3 \times \max(S_{d^3} - X, 0)$$

Overall ‘probability’ = 1

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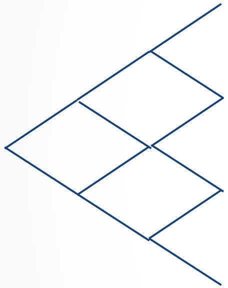
Interpretation

3-step tree



1 → $1 \times q^3 \times \max(S_{u^3} - X, 0)$

3 → $3 \times q^2 (1-q)^{3-2} \times \max(S_{u^2 d} - X, 0)$

3 → $3 \times q^1 (1-q)^{3-1} \times \max(S_{u^1 d^2} - X, 0)$

1 → $1 \times (1-q)^3 \times \max(S_{d^3} - X, 0)$

Value at time = t $\xleftarrow{e^{-r(T-t)}}$ Expected payoff at time = T (maturity)

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The price of a European put option

- Today's spot price is 50
- The risk free rate is 5%
- We want to price a 2-year European put on a stock with exercise price 52

- Use a 2-step model

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The price of a European put option

- Today's spot price is 50
- The risk free rate is 5%
- We want to price a 2-year European put on a stock with exercise price 52

- Use a 2-step model

- u=1.2
- d=0.8
- T = 2
- $\Delta t=1$
- S0 = 50
- X = 52
- r = 5%

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Put option - steps

- 1. Calculate and draw the expected price development of the underlying asset
- 2. Calculate the value of the option at expiry/maturity
- 3. Start at the end nodes and roll back to the start node

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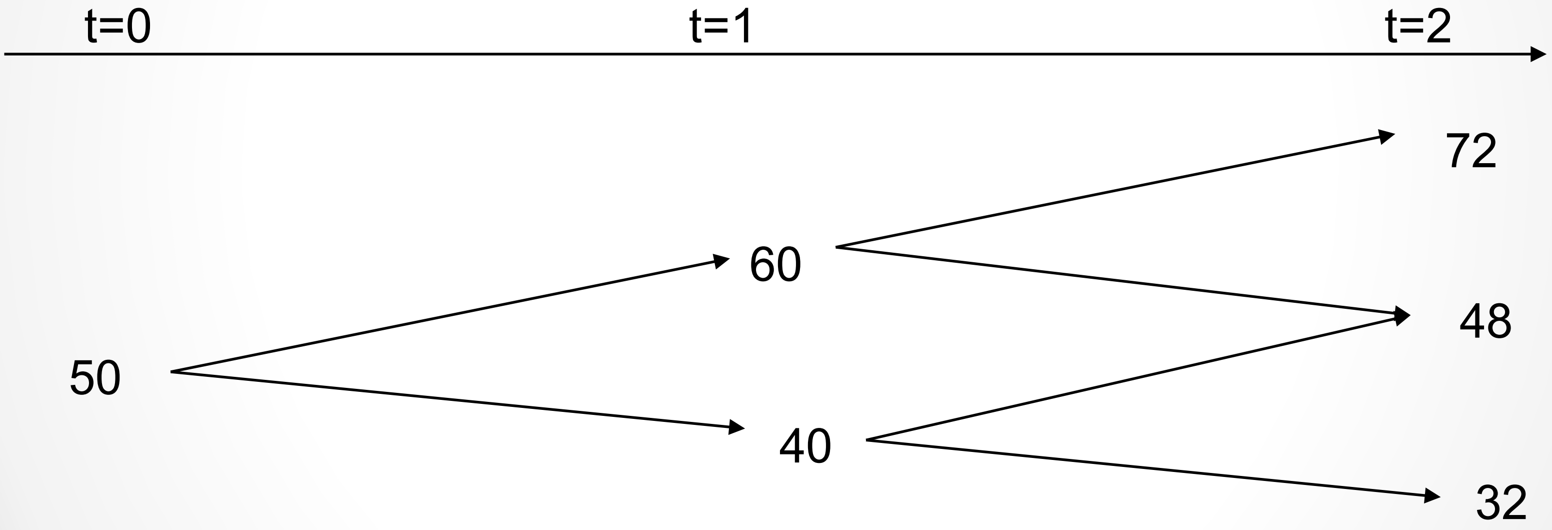
Put option



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Put option

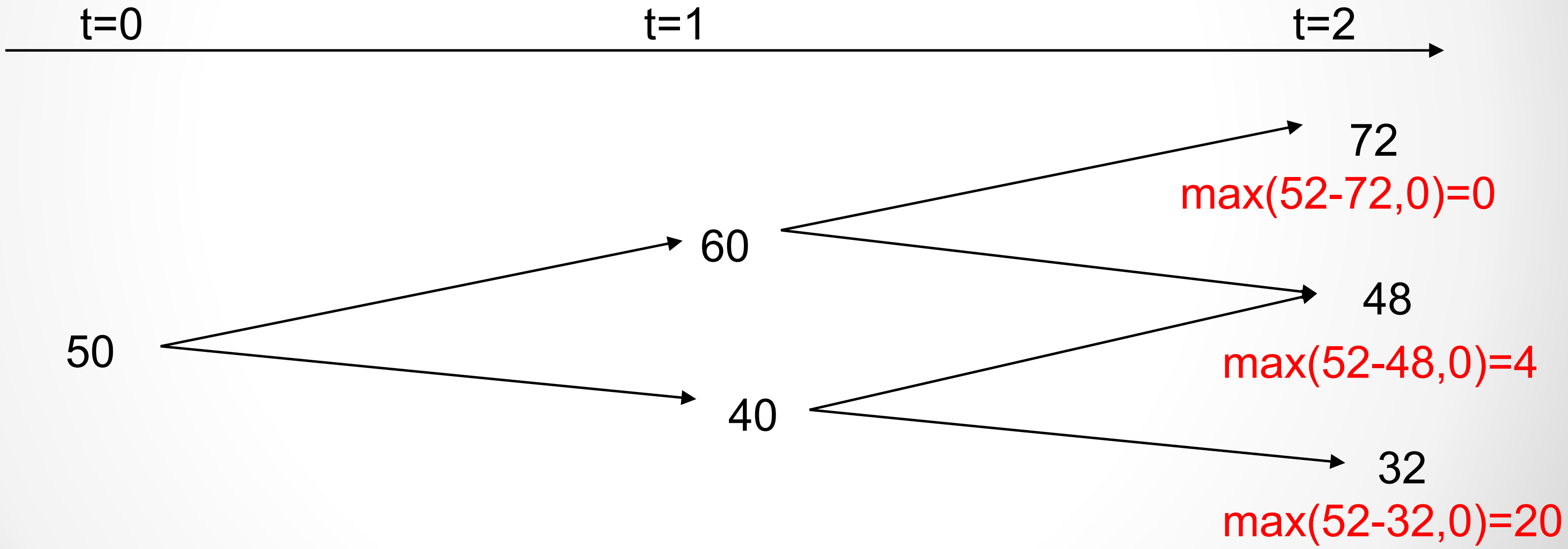
First, calculate the risk neutral probabilities:

$$q = \frac{e^{0.05x1} - 0.8}{1.2 - 0.8} = 0.6282$$

1-q = 1- 0.6282 = 0.3718

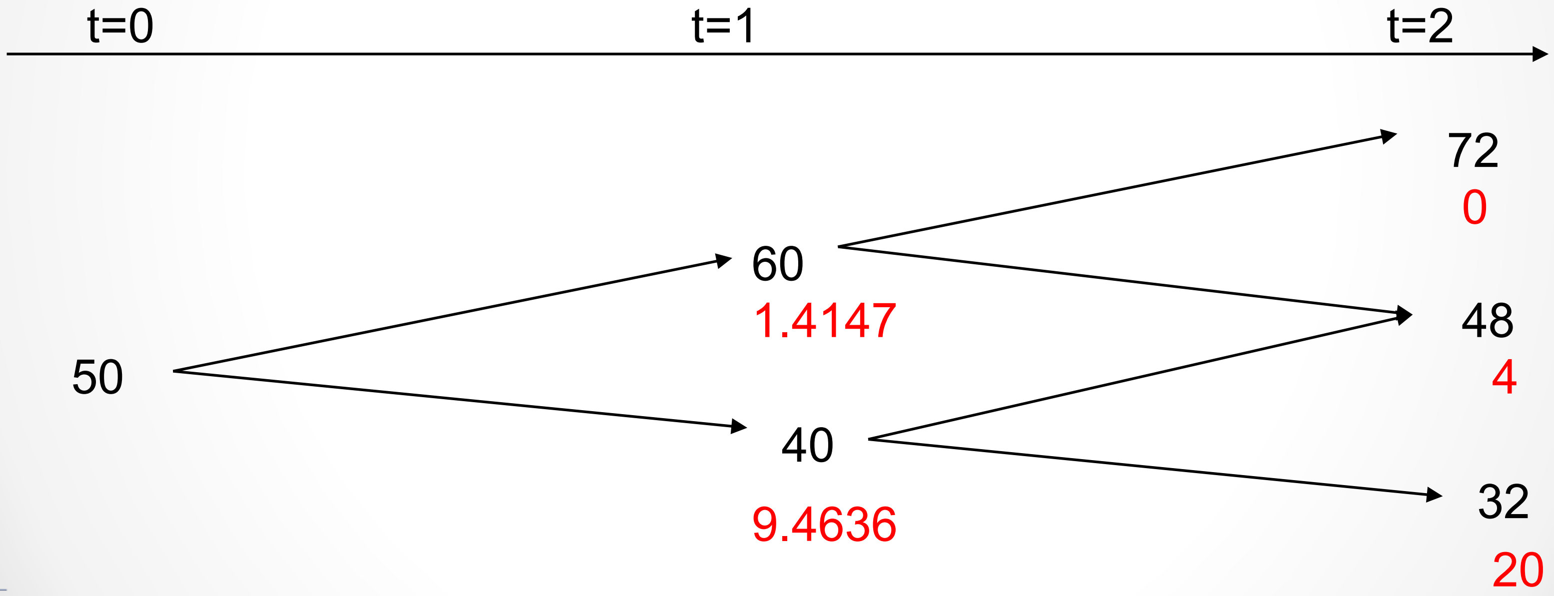
Put option

Value of put at expiry: max (X-S_T,0)



Put option

Value of put at t = 1



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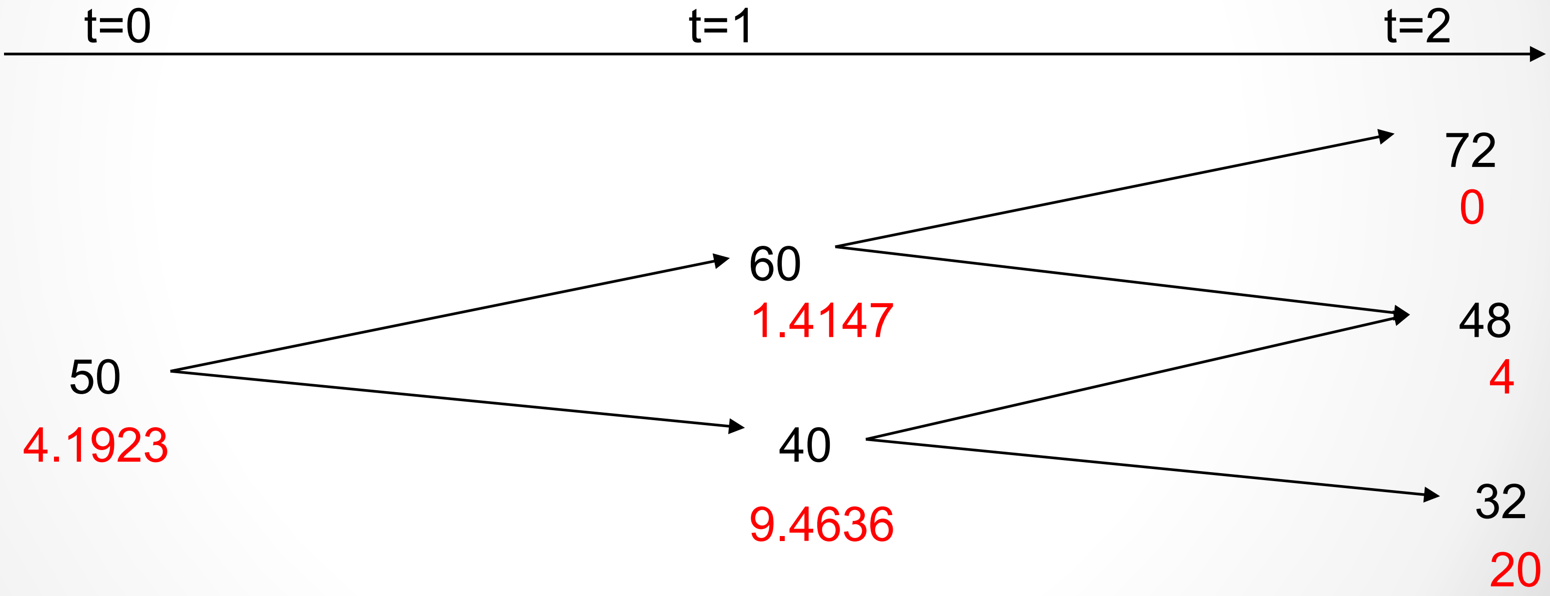
Put option

Value of put at t=0



Put option

Alternative calculation method: (only European options)

$$p_0 = e^{-nr\Delta t}\left[q^2 p_{uu} + 2q(1-q)p_{ud} + (1-q)^2 p_{dd}\right]$$

$$p_0 = e^{-2x0.05x1}\left[0.6282^2 x0 + 2x0.6282x0.3718x4 + 0.3718^2 x20\right]$$

$$p_0 = 4.1923$$

Early exercise: American options

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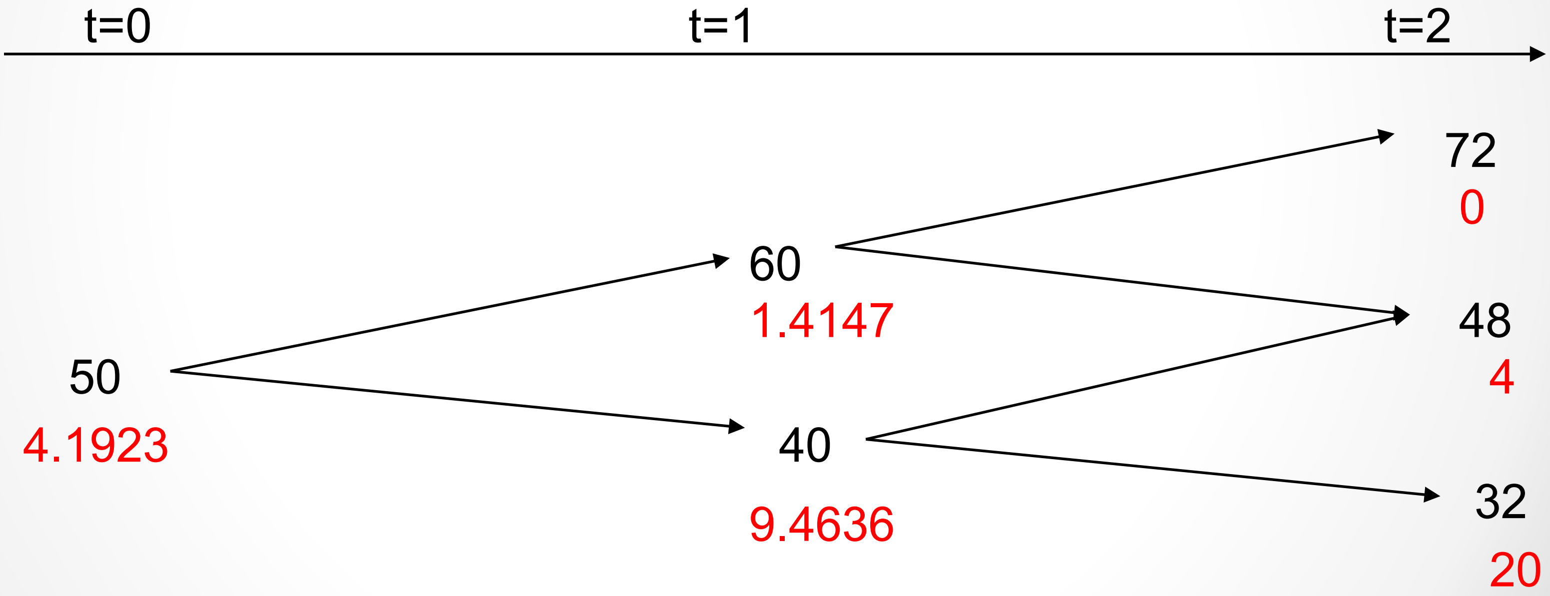
American options

- American options have the possibility of early exercise
- The procedure is to use the same binomial trees as in European options, but you check every node if it is optimal for early exercise
- The value of immediate exercise (intrinsic value)
  - call: max $(S_t\text{-}X,0)$
  - put: max $(X\text{-}S_t,\ 0)$
- This is compared to the option value in the node
- Can calculate the value of early exercise

American put

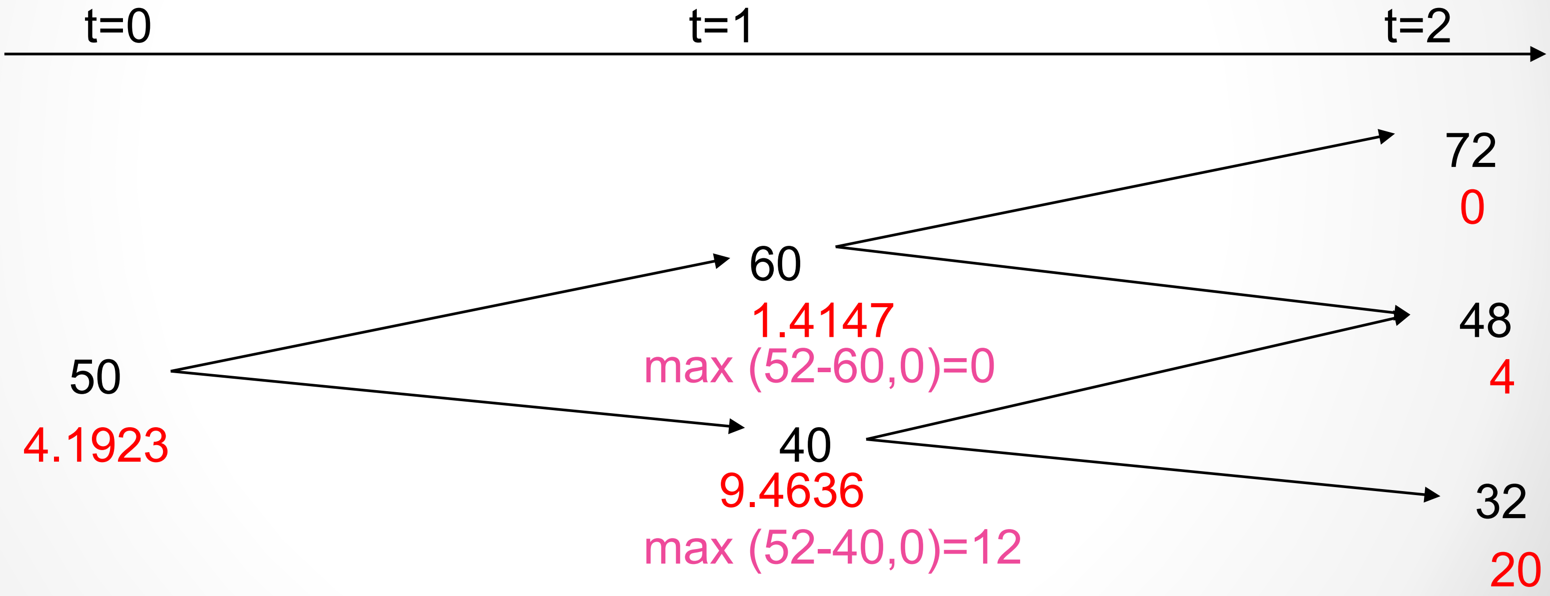
The value of a Europeisk put:



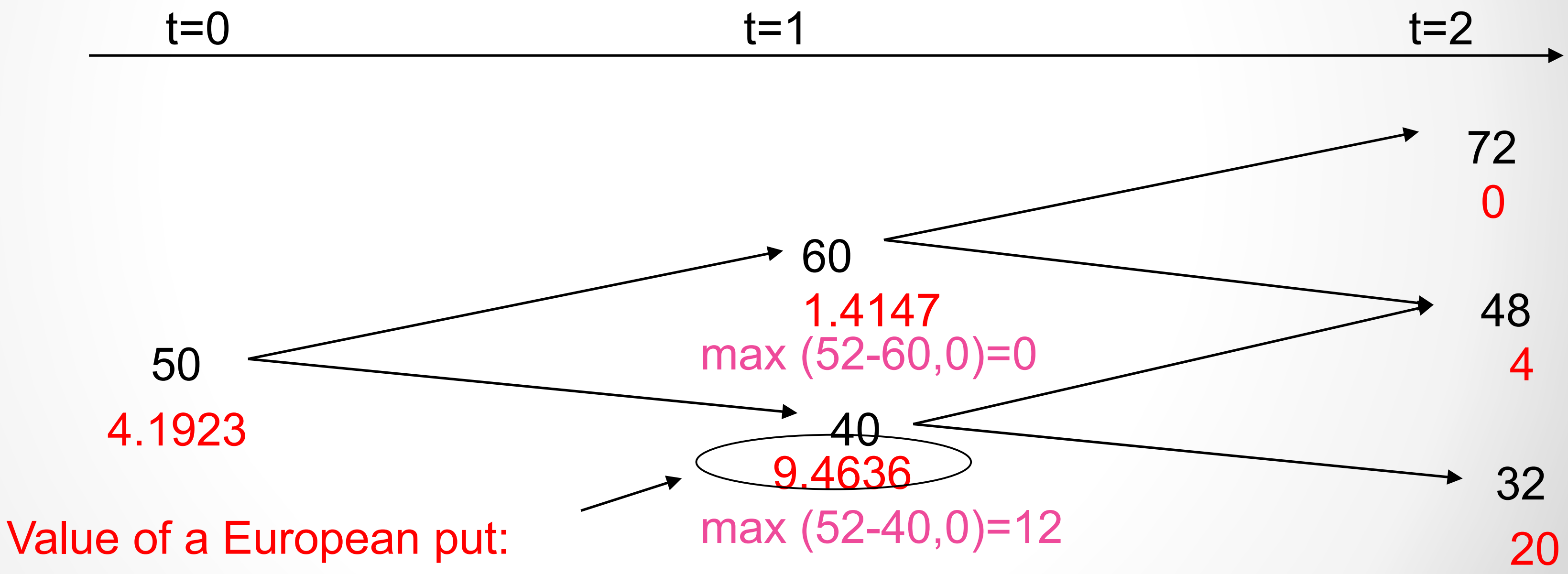
American put

Value of a Europeisk put:

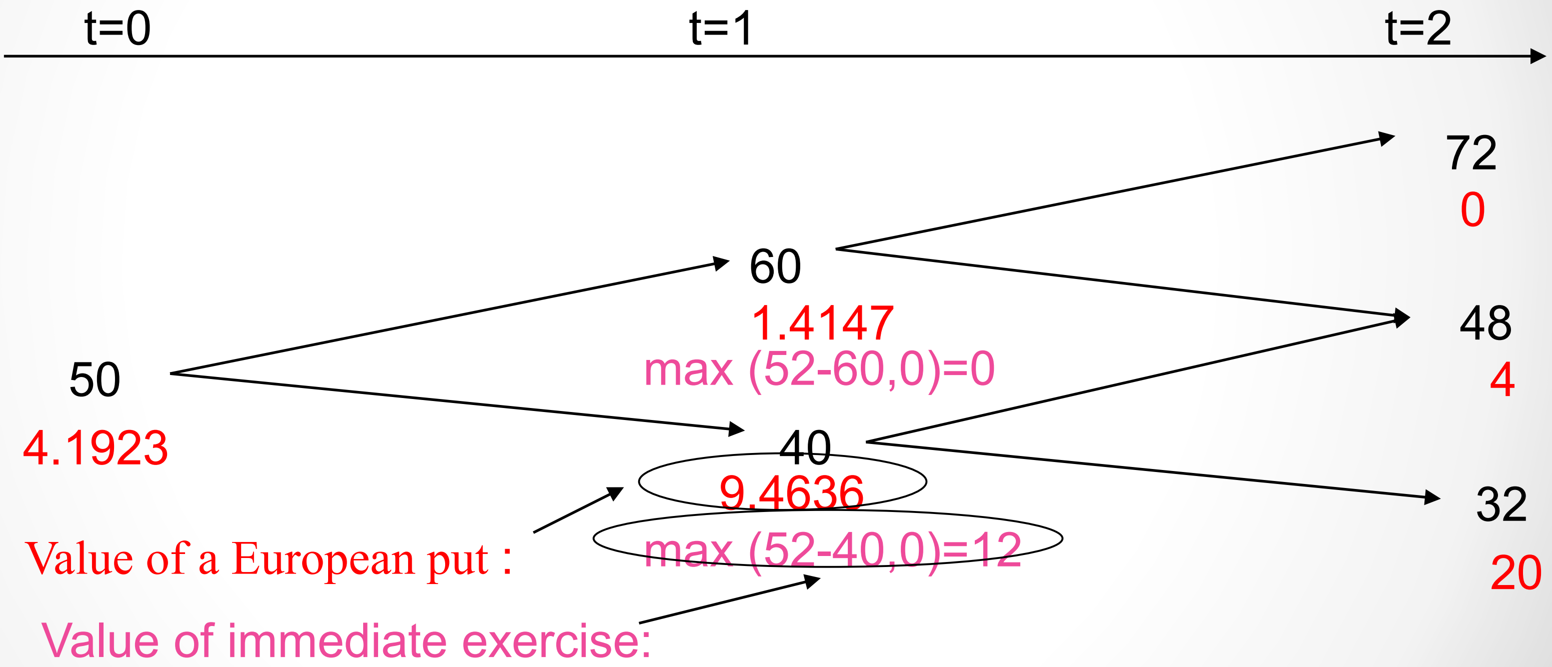
Value of immediate exercise:



American put



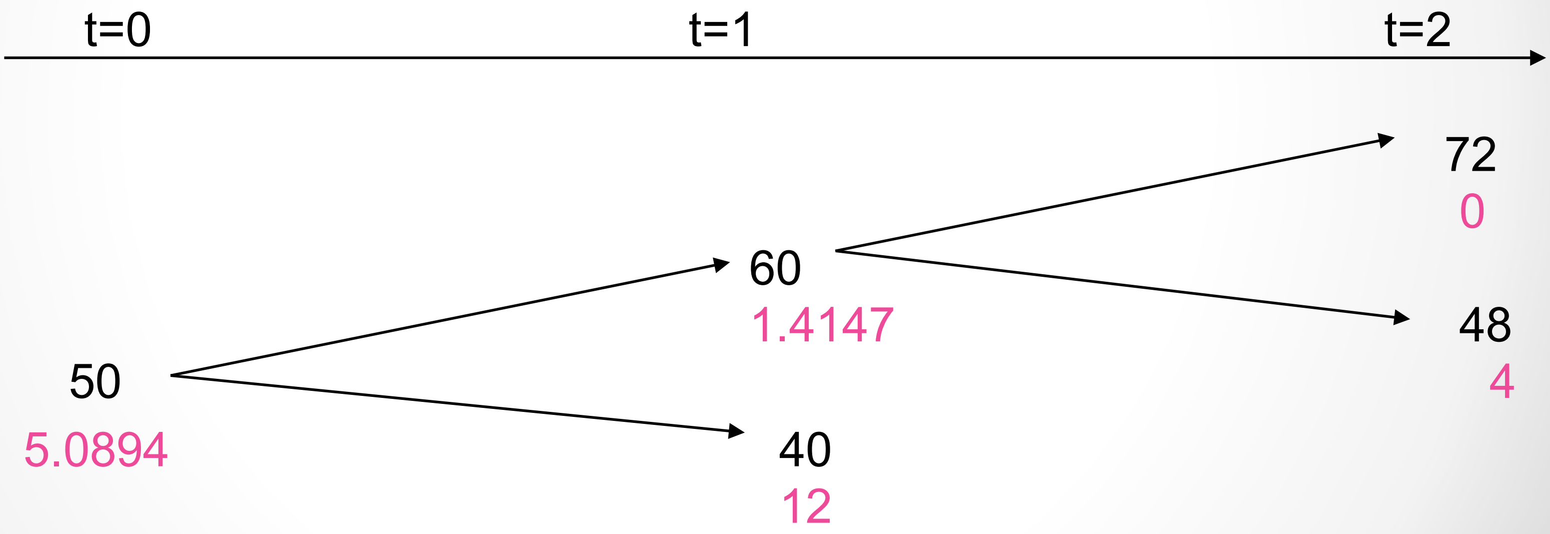
American put



12 > 9.4636 => Immediate exercise is optimal !!

American put

Calculate the option price again, but substitute with the value of early exercise



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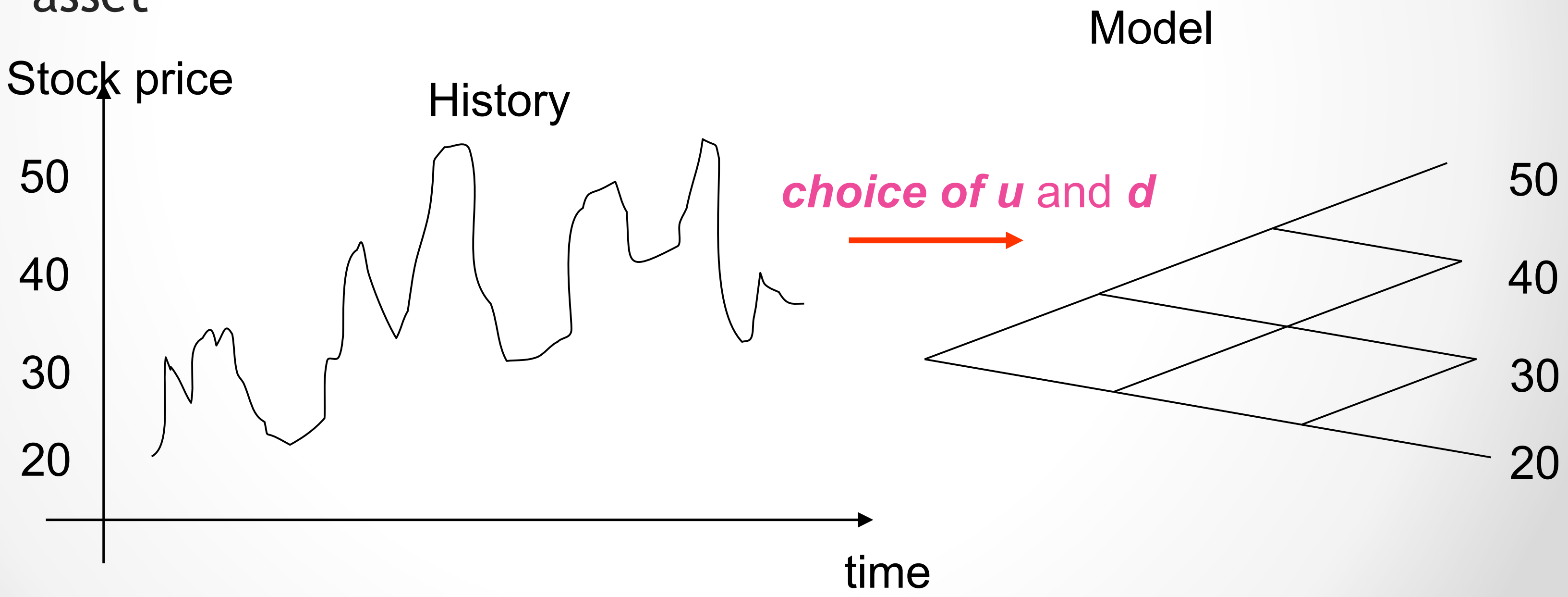
The value of early exercise

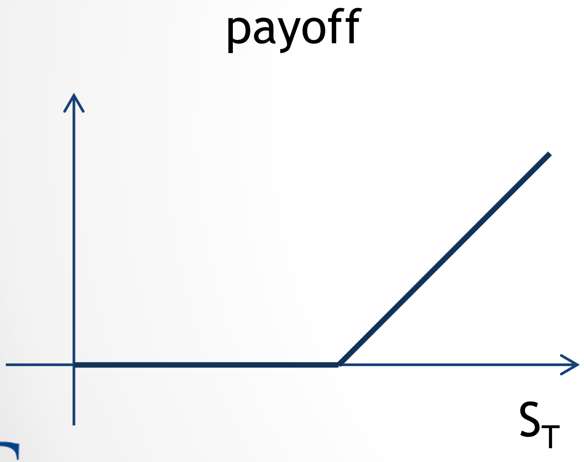
- The value of early exercise =Value of an American option – Value of a European option
- Example:
- The value of early exercise = 5.0894 – 4.1923 = **0.8971**

Matching volatilitet with u and d

- In practice you would select **u** and **d** such that they reflect the price fluctuations (uncertainty, volatility) in the underlying asset



General idea



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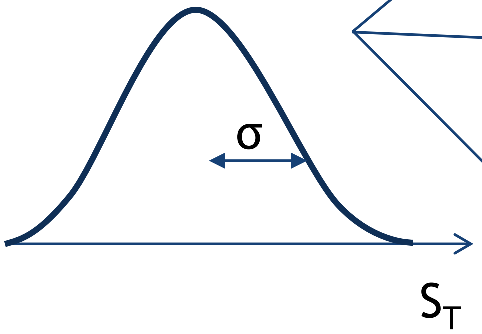
Need to model price behaviour and uncertainty



Closed form (analytical) solutions are derived mathematically

Discretized using time-steps (binomial and trinomial model)

Simulation (Monte Carlo)

$\max(S_T - X, 0)$ → **Valuation**

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Matching volatilitet with u and d

- Cox, Ross, Rubinstein suggested the following relationship

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

- We are using the volatility to determine the magntitude of the up and down factors

- NB! Requires that u =1/d

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Example

- Call option on OBX (OBX 7J400) = 20.50
- (S_0 = 408.74, X= 400, r = 6%, T = 4 weeks)

- Let us price and option and see

- If the volatility of the OBX is 31.5%, and we use a a 2-step model. What is u and d?

$$u = e^{+0.315\sqrt{2/52}} = 1.0637$$

$$d = e^{-0.315\sqrt{2/52}} = 0.9401$$

Example

Option value

0	2 weeks	4 weeks
		62.47
	35.69	
20.10		8.73
	4.39	
		0.00

- Calculated option value = 20.10
- Market quote = 20.50
- The discrepancy can be due to early exercise

Increasing the number of steps

- The 1-step model and the 2-step model is fairly unrealistic
- You can only expect an approximation of the option price by assuming the the stock price only moves 1 or 2 binomial steps during the life of the option
- In practice, the life of the option is often divided into 30 or more steps.
  - Each step represents a binomial change in price
  - With 30 steps ther will be 31 end nodes and 2^{10} or approx. 1 billion possible price paths
- We have to use special software to be able to calculate option values with 30 steps.

Options on other underlying assets

- Options on stocks that pay dividend
- Options on stock indices
- Options on FX
- Options on commodities
- Options on forwards and futures

Options on other underlying assets

- Options on non-dividend paying stocks

$$c_0 = e^{-rT}\left[q \times c_u + (1-q) \times c_d\right] \qquad q = \frac{e^{rT} - d}{u - d}$$

- The price development of the underlying will be affacted by
  - dividend (stocks that pay dividends)
  - Foreign exchange (FX)
- This has to be taken into accounting in the option valuations

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Options on stocks that pay dividends

- Continuous dividend rate, y

$$q = \frac{e^{(r-y)\Delta t} - d}{u - d}$$

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Options on stcok indices

- Continuous dividend rate on index, y

$$q = \frac{e^{(r-y)\Delta t} - d}{u - d}$$

Options on FX

- Foreign exchange rate, r_f

$$q = \frac{e^{(r-r_f)\Delta t} - d}{u - d}$$

Options on forwards and futures

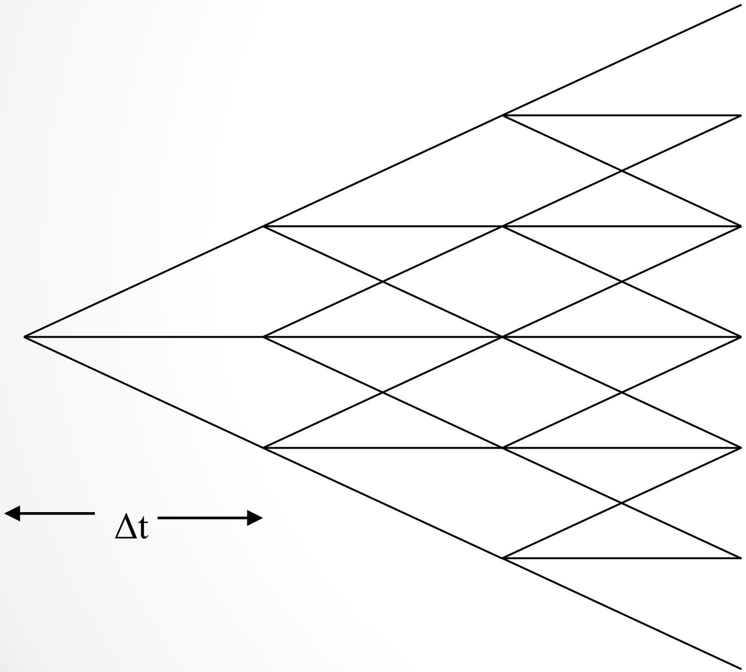
- The expected return on forwards and futures is equal to the continuously compounded risk free rate, r

$$q = \frac{e^{(r-r)\Delta t} - d}{u - d}$$

$$\downarrow$$

$$q = \frac{1 - d}{u - d}$$

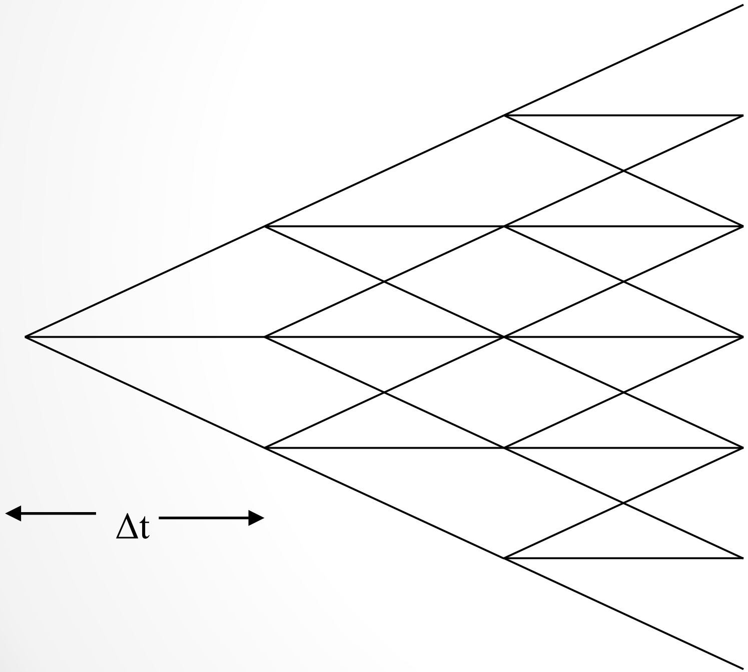
Trinomial trees



Trinomial trees have three possible outcomes compared to binomial trees (two)

1. Up (u)
2. Down (d)
3. Stay the same (m)

Trinomial trees



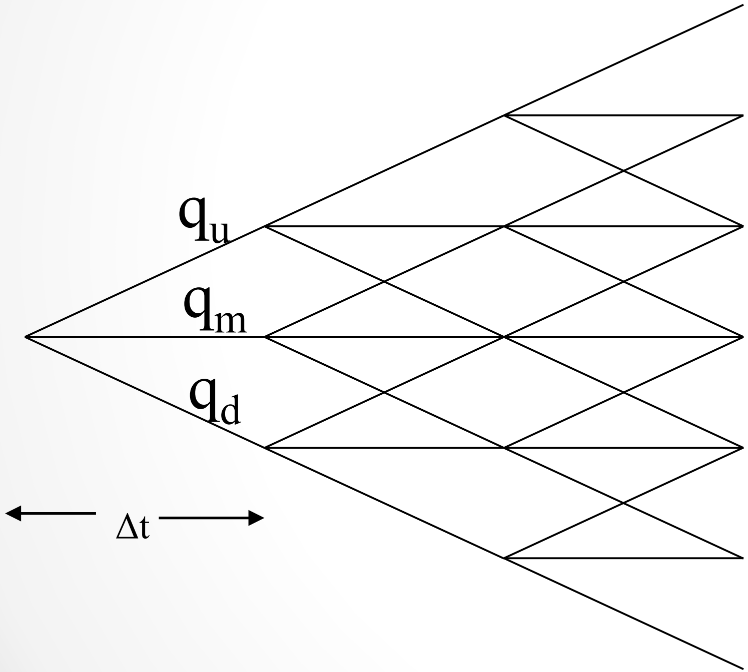
The up (u), down (d)and ‘stay the same’ (m) factors are calculated as

$$u = e^{\sigma\sqrt{3\Delta t}}$$

$$m = 1$$

$$d = \frac{1}{u}$$

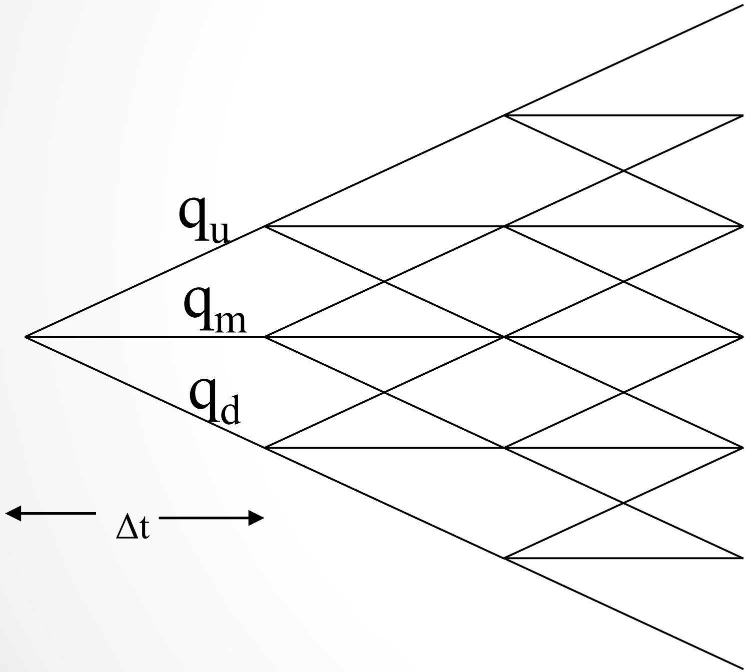
Trinomial trees



With 'probabilities' for each outcome

1. p(Up) = q_u
2. P(Down) = q_d
3. P(Stay the same) = p_m

Trinomial trees



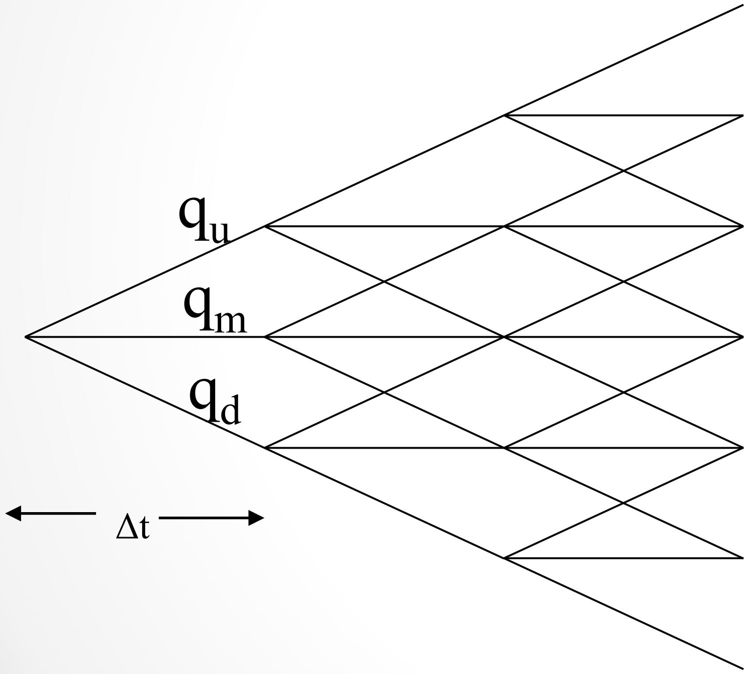
With 'probabilities' for each outcome

$$q_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \delta - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

$$q_m = \frac{2}{3}$$

$$q_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \delta - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

Trinomial trees



- The valuation is analogous to that of binomial trees
  - Start at the end nodes (payoff function)
  - Work backwards recursively
  - At each node calculate the value of exercising and continuing

Value of continuing

$$e^{-r\Delta t}(q_u c_u + q_m c_m + q_d c_d)$$

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Exotic options

- Some exotic options can be valued using binomial trees
- E.g. Barrier options
- Calculate the value of exercising and continuation value
- Example will be given (Knock-out option) later in the course

The Black-Scholes-Merton Model

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The Binomial tree and lognormality

- The binomial tree and lognormality
  - The Random Walk Model
  - Modeling stocks as a Random Walk
  - Continously Compounded Returns
  - Lognormality

- Estimating volatility
  - implied volatility
  - historical volatility

The Random Walk Model

- According to the market Efficiency Theory the price of an asset should reflect all accessible information
- All new information is by definition a surprise
- Future stock prices are therefore uncertain and unpredictable
- According to this theory, the probability of a stock price increase is the same as for a stock price decrease (normal distribution)
- There are 3 problems with this theory
  - Stock prices can become negative (impossible)
  - The size of change should be dependent on how often the stock price changes and the stock price level
  - On average, the return on a stock should be positive

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Continuous compounding

- To avoid these problems, we will use continuous compounding and returns
- Calculate returns from prices: $r_{t,t+h} = \ln(\frac{S_{t+h}}{S_t})$
- Calculate prices from returns: $S_{t+h} = S_t e^{r_{t,t+h}}$
- Continuous returns are additive $r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$
- Prices can never become negative

Examples

- Return (S_t=100, S_{t+h}=110)

$$r_{t,t+h} = \ln(\frac{S_{t+h}}{S_t}) = \ln(\frac{110}{100}) = 0.0953$$

- Prices (S_t=100, $r_{t,t+h}$=0.0953)

$$S_{t+h} = 100e^{0.0953} = 110$$

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Example (2)

- $S_0 = 100,\ S_1 = 105,\ S_2 = 115,\ S_3 = 120$

- Return:

$r_{0,1} = \ln(105/100)$	$= 0.0488$
$r_{1,2} = \ln(115/105)$	$= 0.0910$
$r_{2,3} = \ln(120/115)$	$= 0.0426$
Sum	$= 0.1823$

Check: $r_{0,3} = \ln(120/100) = 0.1823$

Discrete returns are not additive

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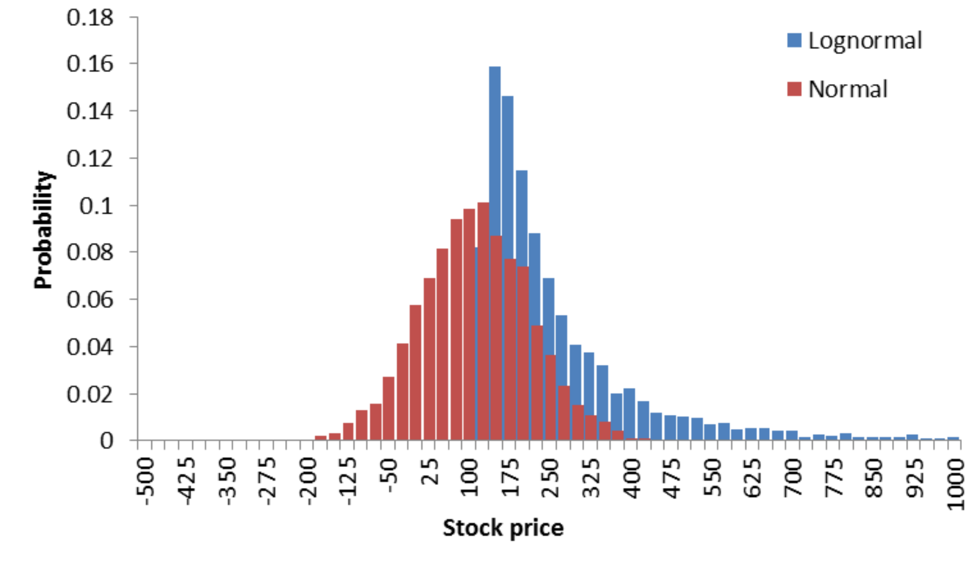
Lognormal distribution

- Stock prices assume to be lognormally distributed
- Log-returns are then *normally* distributed



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Volatility

- The volatility, σ, of a stock is a measure of the uncertainty in the stock price returns
- The volatility of a typical stock is around 15-60%
- Volatility is defined as the standard deviation of log-returns
- Given as an annual size
- Can be calculated from prices with varying granularity
  - hours
  - daily
  - weekly
  - monthly

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Volatility (2)

- Turning volatility with different granularity into a yearly number:

$$\sigma_h = \sigma\sqrt{h}$$

- n = number of time periods per year (granularity)
- h = length of time period (h = 1/n = Δt (!!))
- σ = annual volatility (continuous compounding)

$$\sigma_{week} = \sigma\sqrt{1/52} \qquad \sigma_{month} = \sigma\sqrt{1/12} \qquad \sigma_{daily} = \sigma\sqrt{1/252}$$

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Calculation of volatility

- 1. Implied volatility
  - calculated from option prices
  - Black-Scholes
  - Oslo børs option calculator

- 2. Historical volatility
  - calculated from historical prices
    - Simple average
    - Rolling average
    - EWMA
    - GARCH

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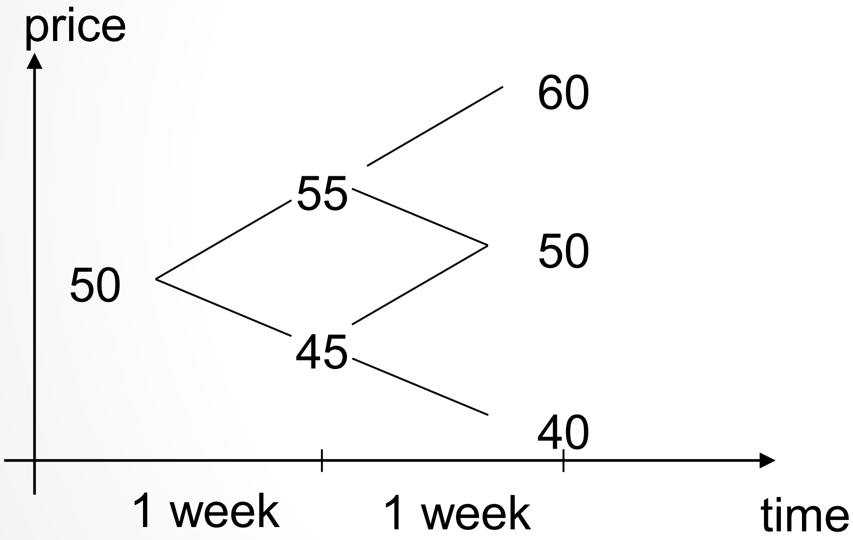
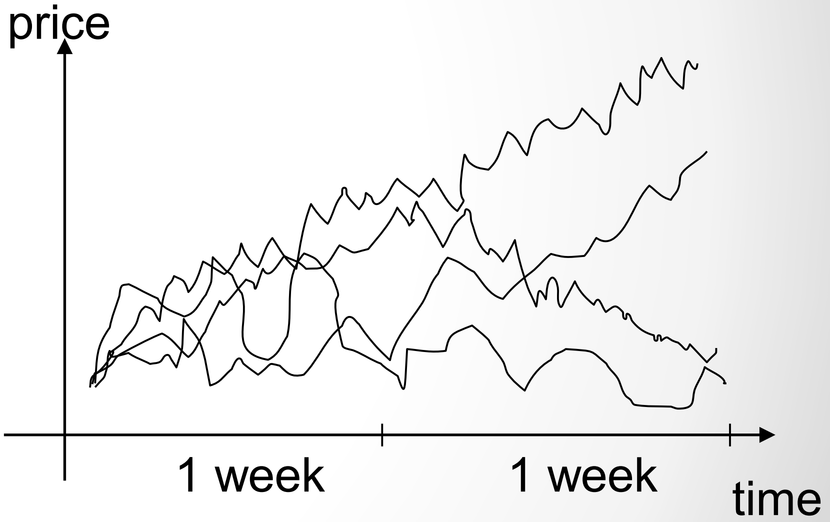
From discrete to continuous time

Discrete time



every time step is 1 week

Continuous time



every time step is less than one second

From discrete to continuous time (2)

- Binomial model

$$E^Q[S_{t+h}] = S_t \underbrace{e^{(r-\delta)h \pm \sigma\sqrt{h}}}$$

expected stock price (risk neutral)

today's stock price

Expected return

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From discrete to continuous time (3)

- Taking logs:

$$\ln(S_{t+h} / S_t) = (r - \delta)h \pm \sigma \sqrt{h}$$

log return

risk free rate

uncertainty (up-move or down-move)

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From discrete to continuous time (4)

- Moving out in time along the binomial tree (that is from time 0 to time T) we can add the binomial uncertainties ($\pm\sigma\sqrt{h}$) together
- When $n \rightarrow \infty$, (or $h \rightarrow 0$), the sum of the binomial random variables will be normally distributed
- In a binomial tree the continuously compounded returns will be (appproximately) normally distributed, and the log returns will be normally distributed

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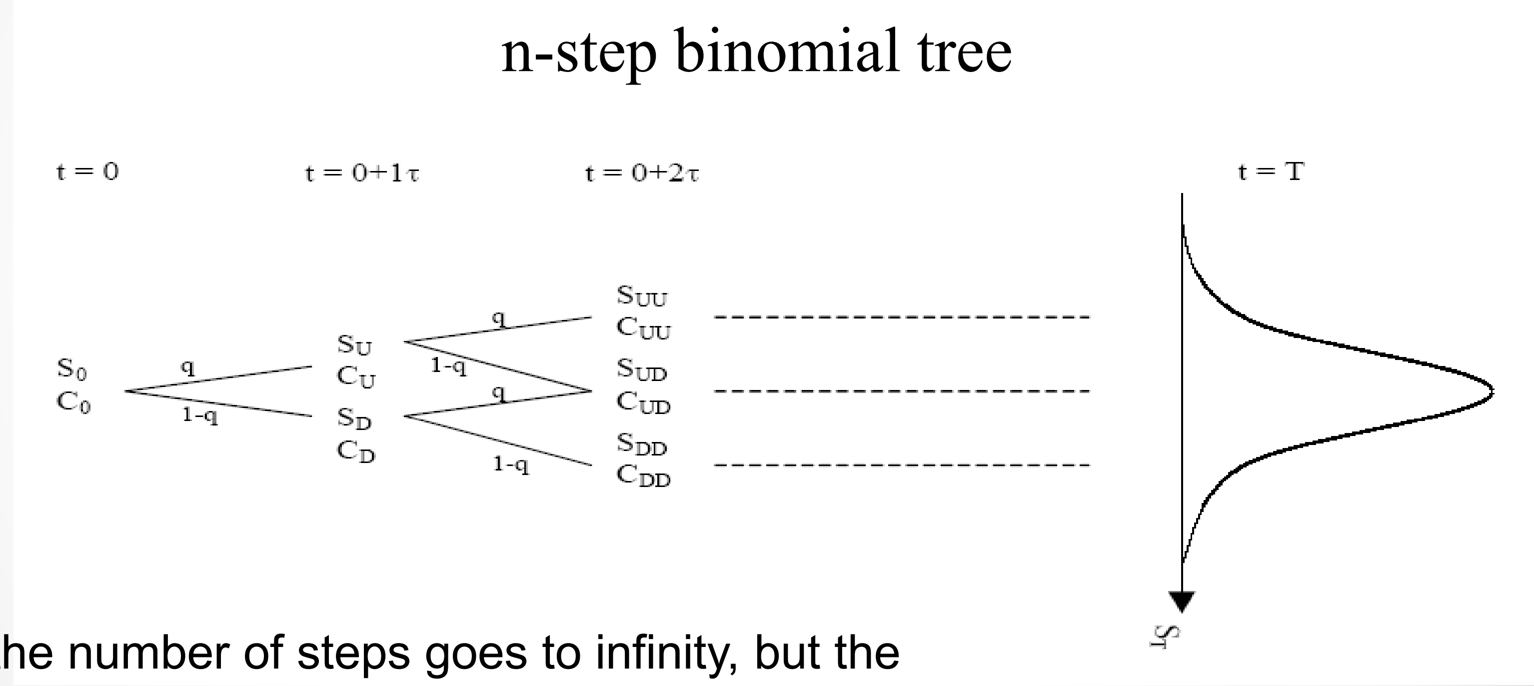
From discrete to continuous time (5)



the number of steps goes to infinity, but the time to maturity is held constant

log normal stock returns

The Black-Scholes formula

- In 1973 Fischer Black and Myron Scholes derived their theoretical option pricing formula

- Black and Scholes' work, in addition to similar work by Robert Merton revolutionised theoretical and practical finance

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Binomial vs Black-Scholes

Antall steg (n)	Binomisk call
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
∞	6.961

← 30 steps (row 50)

← Black-Scholes (row ∞)

Data: S0 = 41, x = 40, vol = 0.30, r = 0.08, T = 1 og δ = 0

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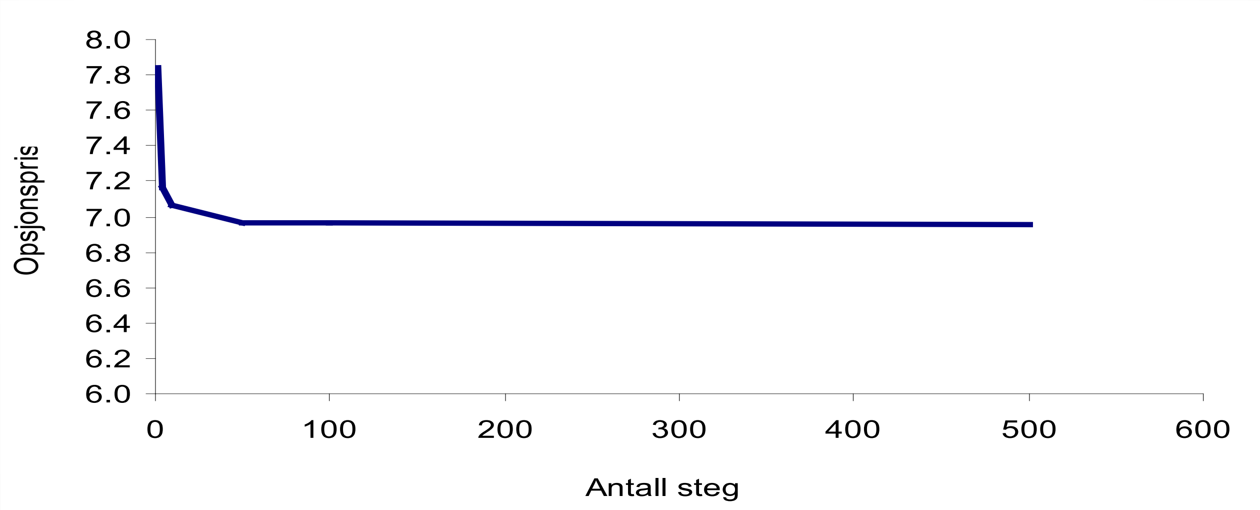
Binomial vs Black-Scholes



Assumptions

- The derivation of the Black-Scholes formula is based on a set of assumptions

- 2 main types of assumptions

- 1. Assumptions about the distribution of prices
    - Continuously compounded returns that are lognormally distributed and independent over time
    - The volatility of log-returns are known and constant
    - future dividends are known and constant

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Assumptions (2)

- 2. Economical assumptions
  - The risk free rate is known and constant
  - No transaction costs or taxes
  - Short sales are free (no costs)
  - It is possible to borrow at the risk free rate

- It is also possible to derive option pricing formulas with stochastic (not constant or deterministic) volatility, dividends and risk free rates

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Call option

- The Black-Scholes option pricing formula for a European call option on a stock that pays dividends (continuous rate) is

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

S_0 = today's stock price
X = strike price
σ = volatility (continuous)
r = risk free rare (continuous)
δ = dividend rate (continuous)
T = time to maturity
N(x) = cumulative normal (probability) distribution function

N(x)

- The function N(.) is the cumulative probability distribution for en standard normal distributed variable
- N(x) is the probability that a variable (that has standard normal distribution, ϕ[0,1]), is less than x

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

Cumulative Normal distribution

Calculation of N(x)

- In Excel you can use NORMSDIST() or NORMSFORDELING()
- We can also use a density distribution table

Tabell for N(x) når x>0

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852

N(0.62) = 0.7324

N(0.6278 = N(0.62) + 0.78[N(0.63)-N(0.62)]

= 0.7324 + 0.78 x (0.7357 – 0.7324)

= 0.7350

Derivation of the Black-Scholes formula (the very short version)

- The value of an option at maturity:

$$c_T = E^Q\left[\max(S_T - X, 0)\right]$$

- The value of an option today:

$$c_0 = e^{-rT} E^Q\left[\max(S_T - X, 0)\right]$$

- Black-Scholes

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

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Example

- S = 41, K = 40, σ = 0.30, r = 0.08, T = 0.25, δ = 0. What is the value of a European call?

- First calculate d_1:

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_1 = \frac{\ln(41/40) + (0.08 - 0 + \frac{1}{2}0.30^2)0.25}{0.30\sqrt{0.25}} = 0.3730$$

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Example (2)

- Then calculate d_2:

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$d_2 = 0.3730 - 0.30\sqrt{0.25} = 0.2230$$

Example (3)

- Then calculate N(d1) and N(d2)
- N(d1) = N(0.3730) = 0.6454
- N(d2) = N(0.2230) = 0.5882

Example (4)

- Then calculate the option price

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

$$c_0 = 41 e^{-0 \times 0.25} 0.6454 - 40 e^{-0.08 \times 0.25} 0.5882 = 3.399$$

B-S: Put option

- Black-Scholes' price formula for a European put on a stock that pays dividends (continuous rate) is:

$$p_0 = Xe^{-rT}N(-d_2) - S_0 e^{-\delta T}N(-d_1)$$

$$N(-d_x) = 1 - N(d_x)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Example

- S = 41, K = 40, σ = 0.30, r = 0.08, T = 0.25, δ = 0. What is the price of a European put?

$$d_1 = \frac{\ln(41/40) + (0.08 - 0 + \tfrac{1}{2}0.30^2)0.25}{0.30\sqrt{0.25}} = 0.3730$$

$$-d_1 = -0.3730$$

$$N(-d_1) = 0.3546$$

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Example (2)

$$d_2 = 0.3730 - 0.30\sqrt{0.25} = 0.2230$$

$$-d_2 = -0.2230$$

$$N(-d_2) = 0.4118$$

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Example (3)

$$p_0 = Xe^{-rT}N(-d_2) - S_0e^{-\delta T}N(-d_1)$$

$$p_0 = 40e^{-0.08\times0.25}0.4118 - 41e^{-0\times0.25}0.3546 = 1.607$$

Put-Call parity

- For European calls and puts (with the same input variables) the following relationship must hold:

$$p_0 + S_0 e^{-\delta T} = c_0 + X e^{-rT}$$

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American options

- The Black-Scholes formula is designed for European options
- Derivation of option pricing formulas for American options is complicated

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Exercises

- Using the Black-Scholes price formulas for put and calls show that:

$$p_0 + S_0 e^{-\delta T} = c_0 + X e^{-rT}$$

- Hint: use only the following formulas

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

$$p_0 = X e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$$

$$N(-d_x) = 1 - N(d_x)$$

Ch. 14: Black-Scholes continued...

- Value options on other underlying assets
  - Stocks that pay dividends
  - Stock indices
  - FX
  - Futures

Stocks that do not pay dividends

$$c_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

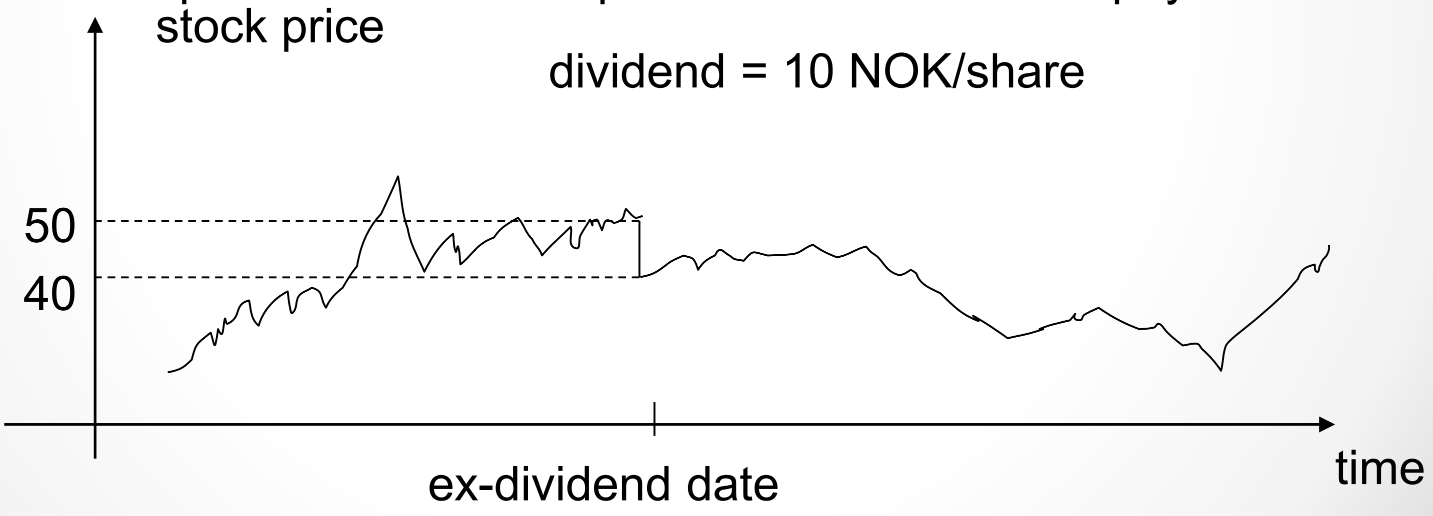
$$d_2 = d_1 - \sigma\sqrt{T}$$

S_0 = today's stock price
X = strike price
σ = volatility (continuous)
r = risk free rate (continuous)
δ = dividend rate (continuous)
T = time to maturity
N(x) = cumulative normal (probability) distribution function

Stocks that pay dividends

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

• Payments of dividends reduces the stock price on the ex-dividend date. The stock price reduction is equivalent to the dividend payment



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Stocks that pay dividends (2)

- The dividend rate, δ, leads to a reduction in the growth rate of the stock price, equivalent to the dividend rate δ.

time = 0	time = T	
S_0	S_T	Stock that pays dividend
$S_0 e^{-\delta T}$	S_T	Stock that *does not* pay dividends

The dividend is reinvested => Larger stock price growth rate

Stocks that pay dividends (3)

- In both cases the probability distribution of the stock price at time T (S_T) is the same

- This means that we can value an option on a stock paying a known dividend rate by reducing today's stock price from S_0 to

  $$S_0 e^{-\delta T}$$

  and then valuing the option as if the stock did not pay a dividend

Stocks that pay dividends (4)

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

S_0 = today's stock price
X = strike price
σ = volatility (continuous)
r = risk free rate (continuous)
δ = dividend rate (continuous)
T = time to maturity
N(x) = cumulative normal (probability) distribution function

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Options on stock indices

- Options on indices can be valued as options on stocks that pay dividends

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

S_0 = today's stock index price
X = strike price
σ = volatility (continuous)
r = risk free rate (continuous)
δ = stock index dividend rate (continuous)
T = time to maturity
N(x) = cumulative normal (probability) distribution function

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Example

- Value a European call on the S&P500 with maturity 2 months. Today's stock index is at 930, the exercise price is 900, risk free interest rate is 8%, volatility 20%, dividend rate is 3%

$$d_1 = \frac{\ln(930/900) + (0.08 - 0.03 + \frac{1}{2}0.20^2)2/12}{0.20\sqrt{2/12}} = 0.5444$$

$$d_2 = 0.5444 - 0.20\sqrt{2/12} = 0.4628$$

$$N(d_1) = 0.7069 \qquad N(d_2) = 0.6782$$

$$c_0 = 930_0 e^{-0.03x2/12} 0.7069 - 900 e^{-0.08x2/12} 0.6782 = 51.83$$

Options on foreign exchange rates (FX)

- Analogous to options on stocks that pay dividends

$$c_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

S_0 = today's stock price
X = strike price
σ = volatility (continuous)
r = domestic risk free rate (continuous)
δ = foreign risk free interest rate (continuous)
T = time to maturity
N(x) = cumulative normal (probability) distribution function

Options on currencies (FX)

- We define S_0 as the spot exchange rate. S_0 is the value of 1 unit of foreign money in norwegian money
- NOK / USD = 5.4
- 1 unit of USD costs 5.4 NOK
- Investment in foreign money => saving money in the bank at the foreign risk free rate, r_f

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Forward price of currencies (1)

- Investment in NOK => saving money in a norwegian bank at the norwegian risk free rate, r
- $B_0 \Rightarrow B_0 e^{rT}$
- Investment in USD => saving money in an American bank at the amerikansk risikofri rente, r_f.
- $G_0 \Rightarrow G_0 e^{r_f T}$

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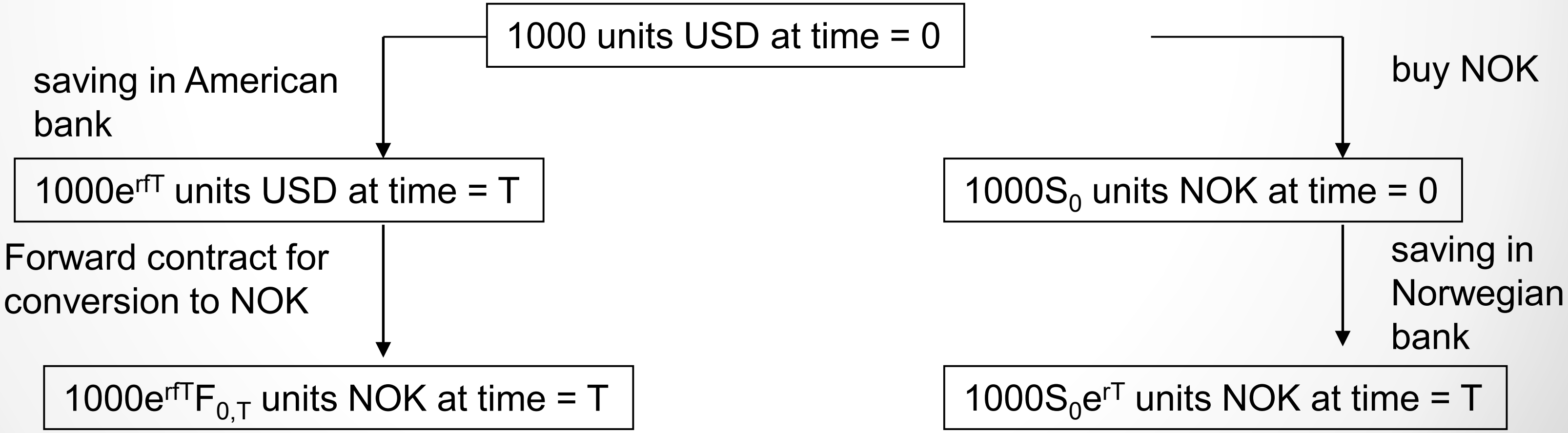
Forward price of currencies (2)

- Two ways of converting 1000 units of foreign currency to NOK at time T
- S_0 = spot exchange rate, $F_{0,T}$ = forward exchange rate



Forward price of currencies (2)

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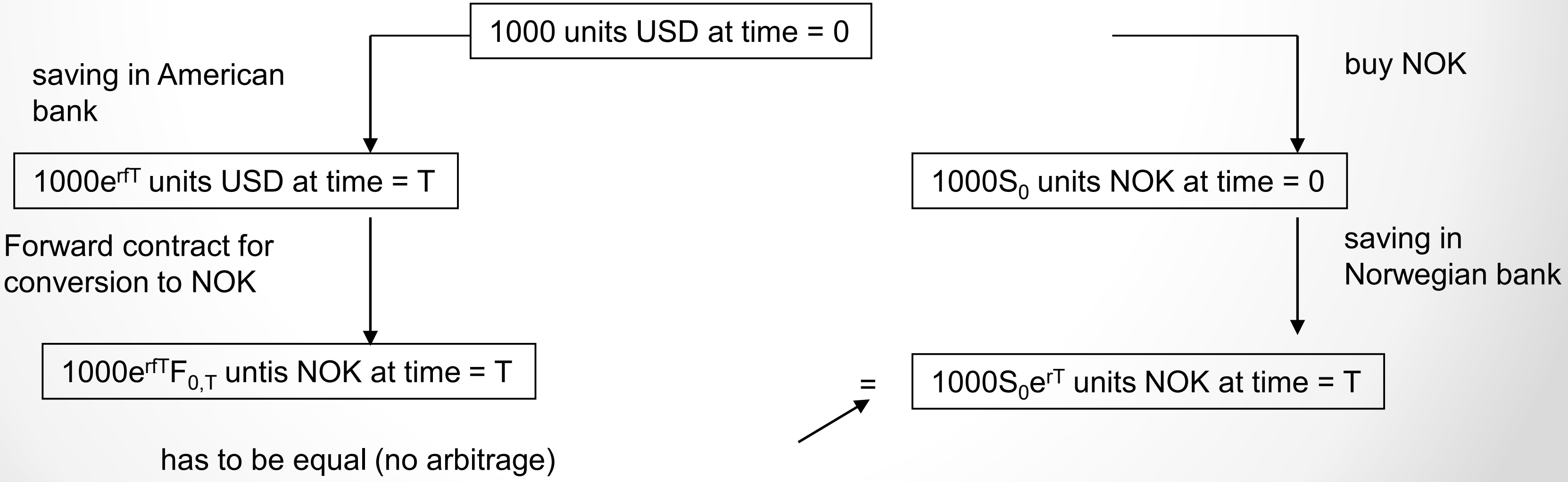

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Forward price of currencies (3)

- This means that:

$$1000e^{r_f T} F_{0,T} = 1000 S_0 e^{rT}$$

- that is, the relationship between $F_{0,T}$ and S_0 is:

$$F_{0,T} = \frac{1000 S_0 e^{rT}}{1000 e^{r_f T}} \Leftrightarrow S_0 \frac{e^{rT}}{e^{r_f T}} \Leftrightarrow S_0 e^{(r - r_f)T}$$

- This is the interest rate parity

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Options on currencies (cont....)

- Foreign currency can be viewed as an investment paying a known ”dividend”
- This ”dividend” is the risk free rate of foreign currency
- If you exchange 100 NOK to USD at an exchange rate of 5 NOK/USD you get 20 USD. This amount is saved in an american bank and grows to $20e^{r_f T}$ during the time T (at a ”dividend rate” of r_f)

Options on currencies (cont....)

- This is analogous to a stock paying dividends
- This means that we can value an option on a foreign currency paying a known dividend rate of r_f by reducing today's stock price from S_0 to $S_0 e^{-r_f T}$

and then valuing the options as if the underlying was a stock that does not pay a dividend

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Options on currencies (cont....)

- Analog til opsjoner på aksjer som betaler utbytte

$$c_0 = S_0 e^{-r_f T} N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

S_0 = today's stock price
X = strike price
σ = volatility (continuous)
r = risk free rare (continuous)
δ = dividend rate (continuous)
T = time to maturity
N(x) = cumulative normal (probability) distribution function

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Example

- Value a European call on british pounds (GBP) with time to maturity 4 months. Today's exchange rate is 1.6000 USD/GBP, the exercise price is 1.6000, the US risk free rate (domestic) is 8%, the british risk free rate (foreign) is 11%, and the volatility is 14.1%

- Answer: 0.043

Options on futures (1)

- The underlying asset is another derivative, a futures contract
- A typical contract is an american call option that requires delivery of an underlying futures contract when the option is exercised
- If the option is exercised, the investor receives a long position in the underlying futures contract plus an amount equal to the last close price minus the strike price
- Equivalent for put: the investor receives a short position in the underlying futures contract plus an amount equivalent to the strike price minus the last close price

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Options on futures (2)

- Example:
  - Assume that today is 15. August and an investor has a September futures call contract on copper with a strike price of 70 cents/kg.
  - 1 futures contract is for 25 tons of copper.
  - Assume that the futures price for copper for delivery in September is 81 cents/kg today.
  - Yesterday's copper futures close price was 80 cents/kg

Options on futures (2)

- If the option is exercised, the investor will receive the following amount:
- 25000 kg x (80 – 70) cents/kg = 2500 USD
- and a long position in a futures contract. If the investor wishes to do so the futures position can be closed, and this will result in the investor receiving:
- 25000 kg x (81 – 80) cents/kg = 250 USD

Options on futures (3)

- The Total payoff from the exercise of the option is 2750 USD (2500 + 250), which is equivalent to
- 25000 x (F – X)

Options on futures (4)

- Generelly: In a risk neutral world a futures price will behave like a stock paying a dividend
- The dividend rate is risk free interest rate, r

Black-76

- Fischer Black developed the following price formula (also known as Black-76) for options on futures contracts

$$c_0 = F_0 e^{-rT} N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

F_0 = today's futures price
X = strike
σ = volatility in the futures price (continuous)
r = risk free interest rate (continuous)
T = time to maturity
N(x) = the cumulative normal distribution function

Black-76

- Fischer Black developed the following price formula (also known as Black-76) for options on futures contracts

$$c_0 = e^{-rT}[F_0 N(d_1) - XN(d_2)]$$

$$d_1 = \frac{\ln(F_0 / X) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

F_0 = today's futures price
X = strike
σ = volatility in the futures price (continuous)
r = risk free interest rate (continuous)
T = time to maturity
N(x) = the cumulative normal distribution function

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Black-76 (put)

- The Black-76 for put options on futures contracts is

$$p_0 = e^{-rT}[XN(-d_2) - F_0 N(-d_1)]$$

$$d_1 = \frac{\ln(F_0 / X) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

F_0 = today's futures price
X = strike
σ = volatility in the futures price (continuous)
r = risk free interest rate (continuous)
T = time to maturity
N(x) = the cumulative normal distribution function

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Example

- Value a European put on a crude oil futures contract. Time to maturity is 4 months, today's futures price is 20 USD/barrel, the exercise price is 20 USD/barrel, the risk free interest rate 9% (annual) and the futures price volatility is at 25%

$$d_1 = \frac{\ln(F_0 / X) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\frac{1}{2}0.25^2 x4/12}{0.25\sqrt{4/12}} = 0.07216$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.07216 - 0.25\sqrt{4/12} = -0.07216$$

$$N(-d_1) = 0.4712 \qquad N(-d_2) = 0.5288$$

$$p_0 = e^{-0.09x4/12}[20x0.5288 - 20x0.4712] = 1.12$$

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