Derivatives and Risk Management in Commodity Markets

Topic 4: Advanced topics in option pricing

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Topics

- Modeling uncertain prices using stochastic differential equations
- Measuring and estimating volatility
- Monte Carlo simulation
  - Stocks
  - Correlated stock prices
  - Commodity prices (mean reverting)

Learning objectives

- Know what stochastic differential equations are and why they are important in option pricing
- Know the difference between constant, deterministic and stochastic processes
- Understand the relationship between real-lif price behaviour and modelled price behaviour

Learning objectives

- Know which stochastic differential equations are relevant for pricing derivatives on energy commodity prices

Learning objectives: Volatility

- Know what implied volatility is
- Know what ‘volatility smiles’ are
- Know why the volatility smile is the same for calls and puts
- Explaining the shape of the volatility surface after 1987 (understand the reasons for the smile and the skew)
- What we mean by the volatility term structure

Learning objectives: volatility

- Know three methods for estimating volatility
  - simple approach
  - Exponentially weighted moving average (EWMA) model
  - Generalised AutoRegressive Conditional Heteroscedasticity (GARCH) model

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Stochastic Differential Equations

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What's the point of SDE's?

- To price options we need to be able to describe the behaviour (i.e. direction over time and uncertainty) of the prices of the underlying asset
- A mathematical representation of the price behaviour
- Key elements: direction and uncertainty

What's the point of SDE's?

$$S_t = f(time) + f(uncertainty)$$

Direction
('drift')

Variation around
the drift element ('stochastic')

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Types

Price level $S_t = f(t) + f(\varepsilon)$

Price change (returns): discrete $\Delta S_t = f(\Delta t) + f(\Delta \varepsilon)$

Price change (returns): continuous $dS_t = f(dt) + f(d\varepsilon)$

Stochastic processes

- Definition:
  - Constant variable
  - Deterministic variable
  - Stochastic variable
- A constant variable has the same value irrespective of time

$$S_t = c$$

- The value of a deterministic variable is a function of time

$$S_t = f(t)$$

- The value of a stochastic variable changes over time in an uncertain way

$$S_t = f(t, \varepsilon)$$

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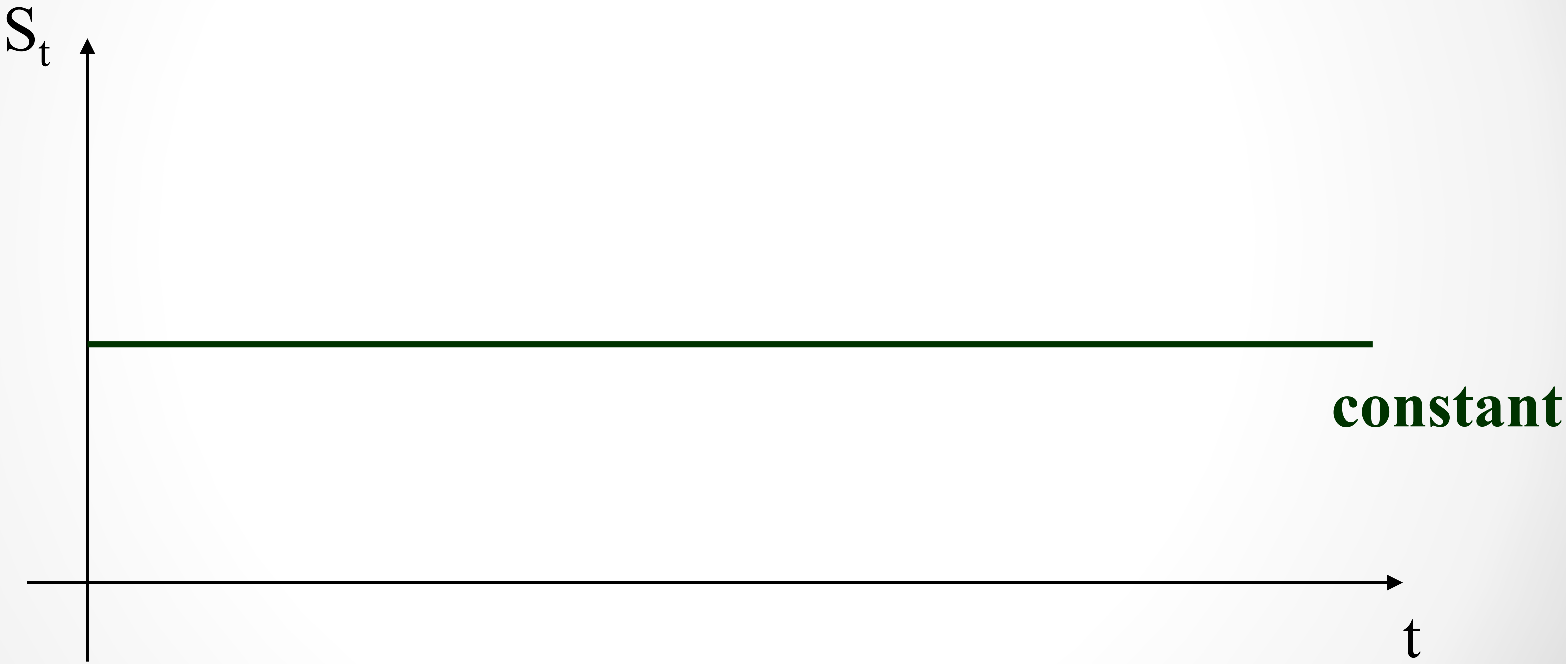
Stochastic processes



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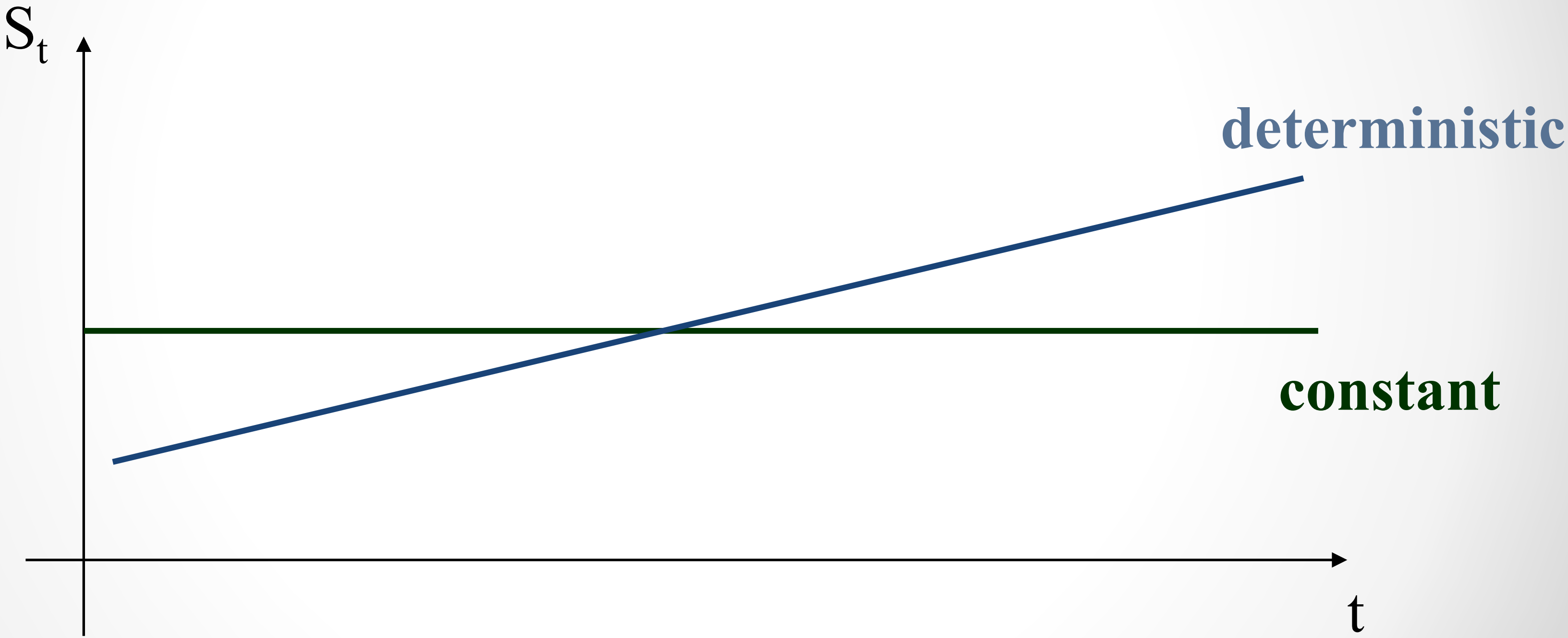
Stochastic processes



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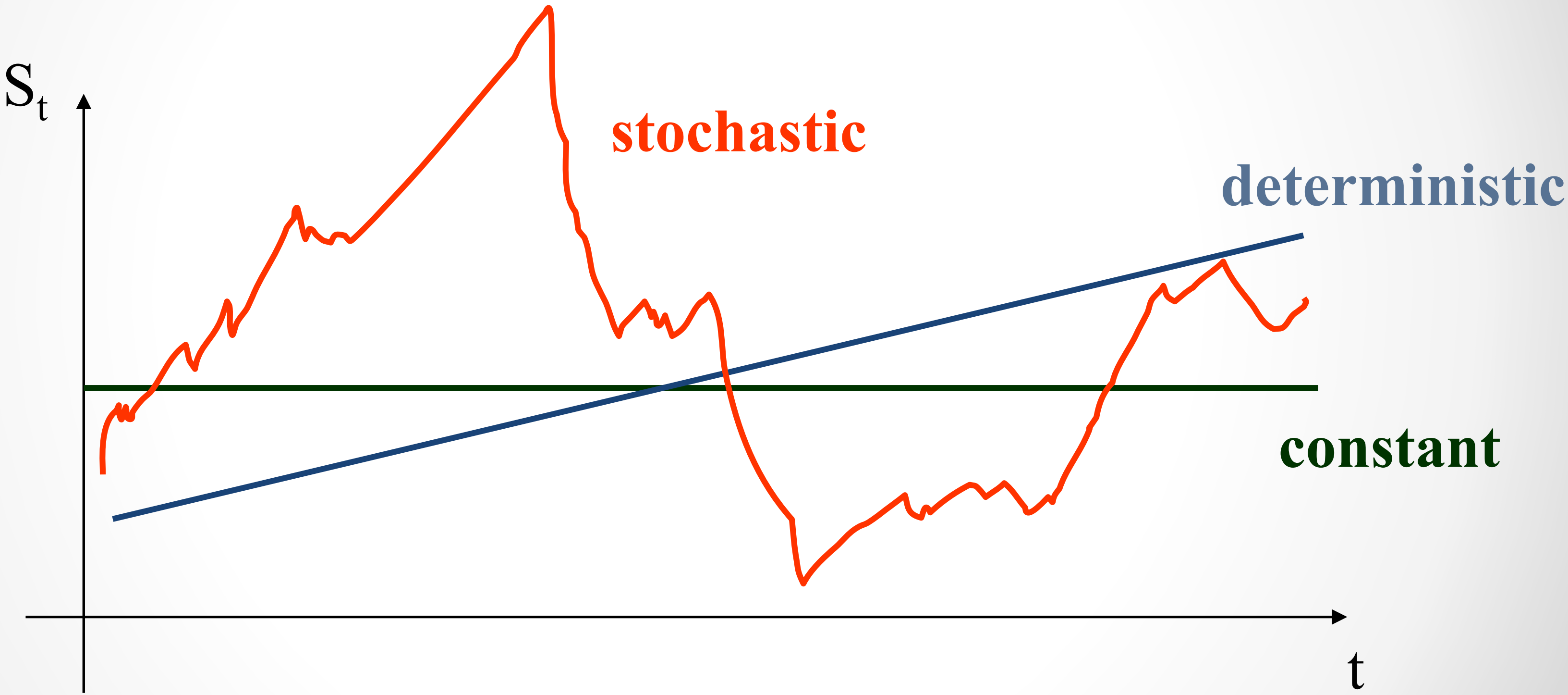
Stochastic processes



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Discrete vs continuous

- 2 classifications of stochastic processes
  - Discrete time
  - Continuous time
- Discrete time
  - The value of a variable can change only at certain fixed points in time
- Continuous-time
  - The value of a variable can change at any time
- We model the price of the underlying asset using a stochastic price process, either discrete or continuous

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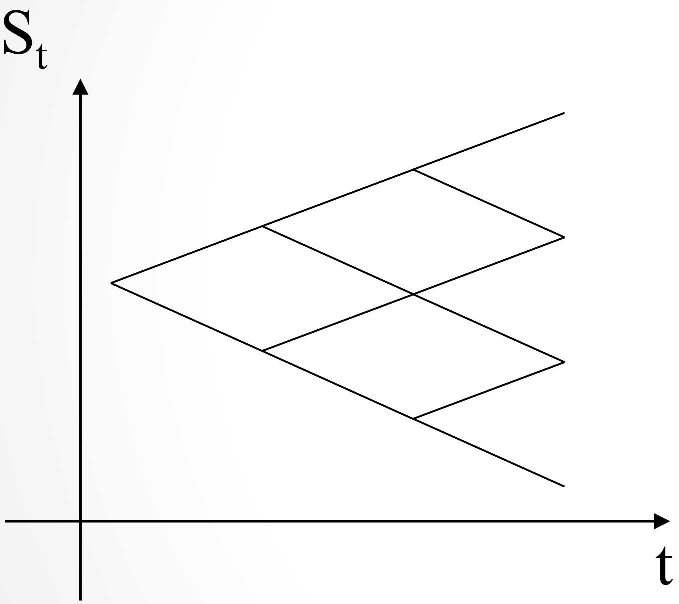
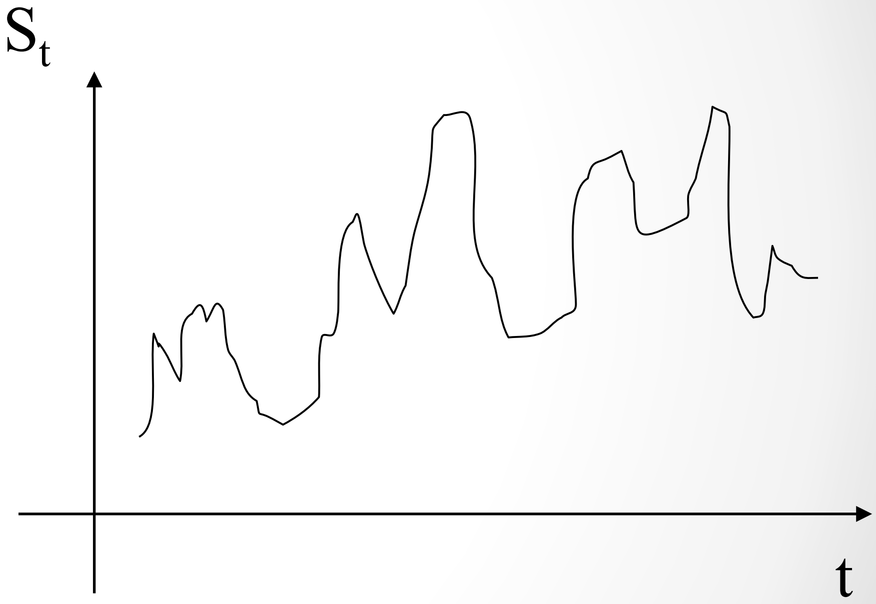
Discrete vs continuous



Discrete timesteps



Continuous timesteps

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Pricing options in continuous time

- Pricing options in discrete time
  - binomial trees
  - trinomial trees

- Pricing options in continuous time
  - Black-Scholes (closed form (analytic) solution)
  - Monte Carlo simulation

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Pricing options in continuous time

- Stochastic price processes (modelling uncertainty)
  - The Markov property
  - Wiener process
- Continuous-time stochastic processes (when Δt ->0)
  - Generalised Wiener
  - Itô process
  - Geometric brownian motion(GBM): the price process for stocks

Simple model

Realistic model for stocks

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SDE for stocks

We want our final model for stock prices to be on this form (e.g. capture both direction and uncertainty)

$$S_t = f(time) + f(uncertainty)$$

How do we develop *f(time)* and *f(uncertainty)*?

That is the topic over the next slides

The Markov property

- The Markov property is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future
  - the past history is irrelevant
    - Fundamental valuation
  - the probability distribution of a price at any point in time is not dependent on the particular path followed by the process in the past

- Stock prices are assumed to follow a Markov process
  - consistent with the weak form of market efficiency

Stochastic time stochastic processes

- Consider a variable (e.g. a price or change in price) that follows a Markov stochastic process
- Suppose the current value is 10 and the change in its value during 1 year is $\phi(0,1)$
- $\phi(\mu,\sigma)$ is a probability distribution with mean μ and standard deviation σ
- What is the probability distribution of the change in the value of the variable during 2 years?
  - The sum of the distributions (since they are independent)

Stochastic time stochastic processes

- Mean
  - 1 year mean = u
  - 2 year mean = u + u = 2u
  - 2 year mean = 0 + 0 = 2x0 = 0
- Standard deviation (square root of variance)
  - 1 year standard deviation = s
  - 1 year variance = s^2
  - 2 year variance = $s^2 + s^2 = 2s^2$
  - 2 year standard deviation = $\sqrt{2s^2} = \sqrt{2 \times 1^2} = \sqrt{2}$
- 2 year distribution: $\phi(0, \sqrt{2})$

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Stochastic time stochastic processes

- 2-year distribution: $\phi(0, \sqrt{2})$
- Generalisation: $\phi(0,\sqrt{\Delta t})$
- The variance is additive, the standard deviation is not
- In terms of:

$$S_t = f(time) + \underbrace{f(uncertainty)}_{\sim\phi(0,\sqrt{\Delta t})}$$

(Markov process)

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Wiener process

- A Wiener process is a **particular type of Markov process** with a mean of 0 and a standard deviation of 1 (per year)
  - Also referred to as Brownian Motion

- A variable W follows a Wiener process if it has the two following properties:

1. The change ΔW during a small period of time Δt is:

$$\Delta W = \varepsilon \sqrt{\Delta t} \qquad \varepsilon \sim \phi(0,1)$$

Wiener process

2. The values of ΔW for any two different short intervals of time, Δt, are independent

- Mean of ΔW $= 0$
- Standard deviation of ΔW $= \sqrt{\Delta t}$
- Variance of ΔW $= \Delta t$

Wiener process

- Short period of time, Δt:

$$\Delta W = \varepsilon \sqrt{\Delta t} \qquad N = \frac{T}{\Delta t}$$

- Longer period of time, T:

$$W(T) - W(0) = \sum_{i=1}^{N} \varepsilon_i \sqrt{\Delta t}$$

Mean of [W(T)-W(0)] $= 0$

Standard deviation of [W(T)-W(0)] $= \sqrt{T}$

Variance of [W(T)-W(0)] $= n\,\Delta t = T$

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Wiener process

$$S_t = f(time) + f(uncertainty)$$

$$\downarrow$$

$$\Delta W = \varepsilon \sqrt{\Delta t}$$

$$\varepsilon \sim \phi(0,1)$$

(Wiener process)

ΔS (discrete change in price) and dS (continuous change in price)

- Small changes in time:

$$\Delta x = a\Delta t$$

- In the limit (as Δt ->0)

$$dx = a \cdot dt$$

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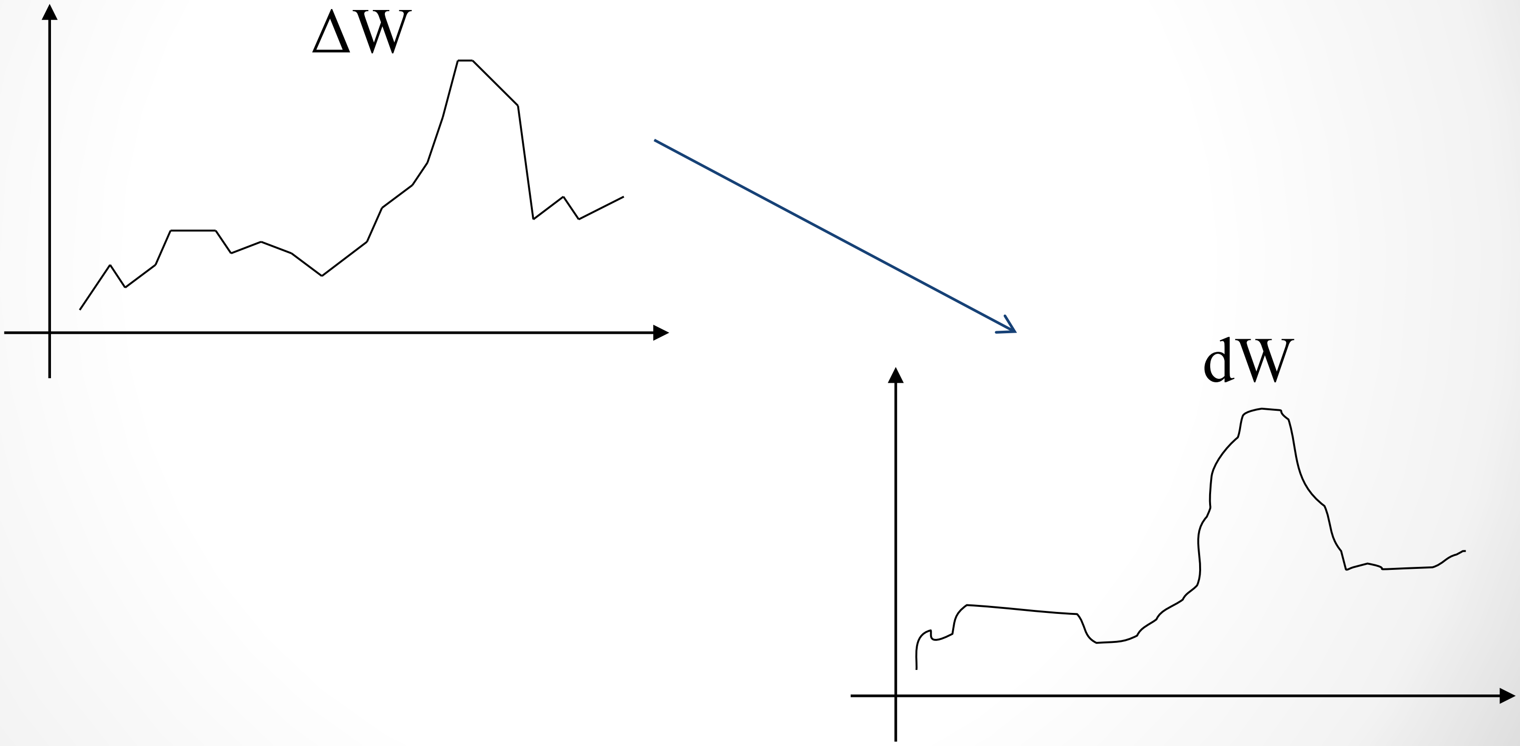
ΔW and dW



Generalized Wiener process

- Drift rate: the mean change per unit time for a stochastic process
- Variance rate: the variance per unit of time

- Wiener process
  - Drift rate = 0
  - Variance rate = 1

Generalized Wiener process

- A Generalised Wiener process for the variable S:

$$dS = \underbrace{\mu dt}_{\text{drift term}} + \underbrace{\sigma dW}_{\text{stochastic term}}$$

μ and σ are constants

- This means that:

$$S_T \approx S_0 + \mu T + \varepsilon\sigma\sqrt{T}$$

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Generalised Wiener process

$S_t = f(time) + f(uncertainty)$ Price process (level)

$dS_t = f(dt) + f(dW)$ Price process (change/ price return)

$dS_t = udt + \sigma dW$ Generalised Wiener (continous time)

$S_T \approx S_0 + \mu T + \varepsilon\sigma\sqrt{T}$ Generalised Wiener (integrated over time T)

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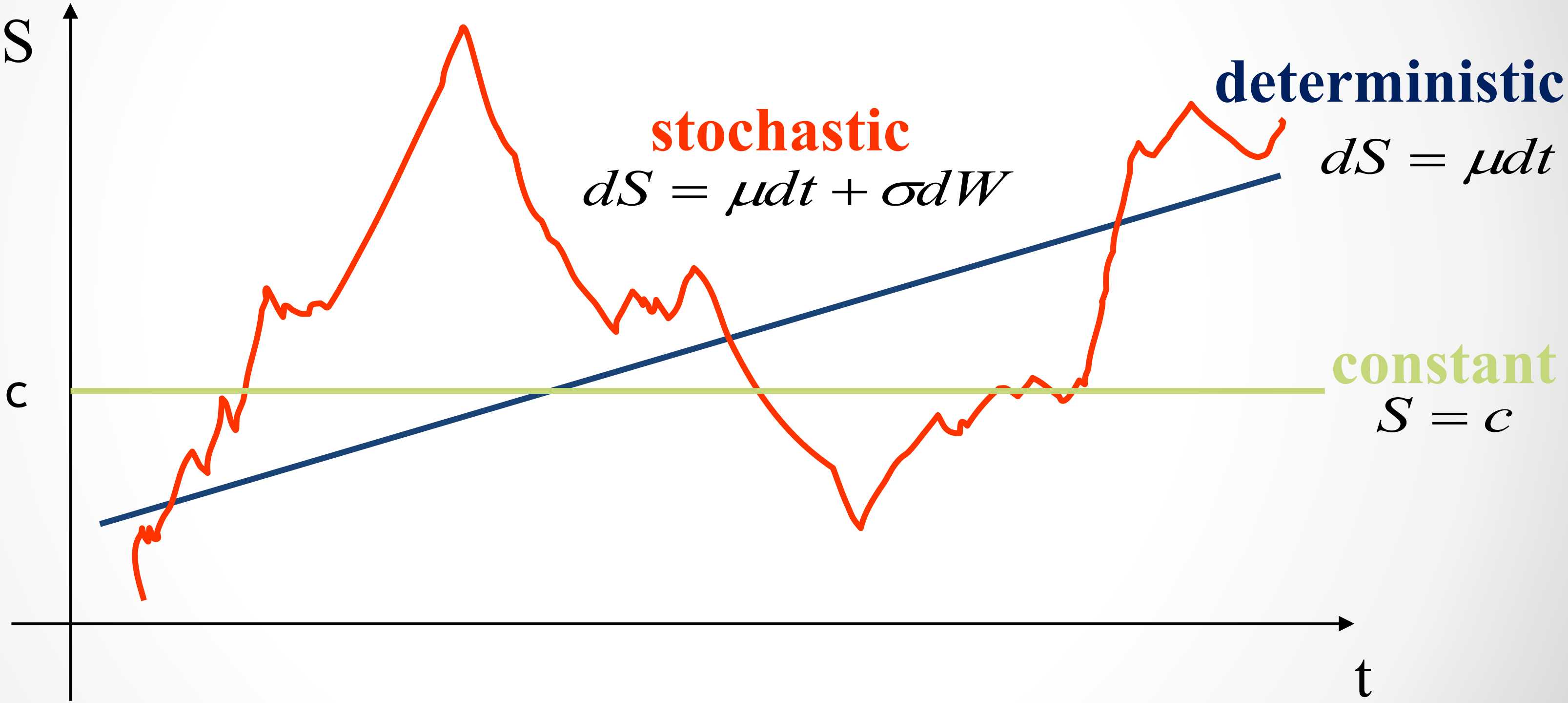
Generalized Wiener process



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Itô process

- Generalised Wiener process:
  - μ and σ are functions of only time

$$dS = \mu(t)dt + \sigma(t)dW$$

- Itô process:
  - μ and σ are functions of both time and the underlying variable S

$$dS = \mu(S,t)dt + \sigma(S,t)dW$$

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Stock price process

- Does a stock price follow a generalised Wiener process?
- Fails to capture key aspects of stock prices
  - the expected return on a stock is independent of the stock's price
    - expected return: μ/S should be constant: $(St-S_0)/S_0$
- A more realistic stock price process is:

$$dS = \mu S dt + \sigma S dW \Leftrightarrow \frac{dS}{S} = \mu dt + \sigma dW$$

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Geometric Brownian Motion (GBM)

- Stock price process

$$\frac{dS}{S} = \mu dt + \sigma dW$$

- μ = expected return on stock
- σ = volatility of stock

- This price process is called Geometric Brownian Motion (GBM)

Geometric Brownian Motion (GBM)

- Stock price process (continuous time)

$$\frac{dS}{S} = \mu dt + \sigma dW$$

- Stock price process (discrete time)

$$\Delta S = \mu S \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

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Geometric Brownian Motion (GBM)

- Stock price process (continuous time)

$$\frac{dS}{S} = \mu dt + \sigma dW$$

- Stock price process (discrete time)

$$\underbrace{\Delta S}_{\text{change in spot price}} = \underbrace{\mu}_{\text{drift}} S \underbrace{\Delta t}_{\text{time step}} + \underbrace{\sigma}_{\text{magnitude ('size of 1 std')}} S \underbrace{\varepsilon}_{\text{Uncertainty ('number of standard deviations')}} \sqrt{\Delta t}$$

The principles of Monte Carlo simulation

Monte Carlo simulation

- We can use the discrete version to simulate prices

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

- We need
  - Expected return: μ
  - Expected volatility: σ
  - Random number generator to create ε

$$\varepsilon \sim \phi(0,1)$$

Monte Carlo simulation

- Suppose
  - the current stock price is 20
  - the expected return from a stock is 14% (per year)
  - the volatility in returns is 20% (per year)
  - time steps (Δt) are in 0.01 years

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

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Monte Carlo simulation

- Suppose
  - the current stock price is 20
  - the expected return from a stock is 14% (per year)
  - the volatility in returns is 20% (per year)
  - time steps (Δt) are in 0.01 years

$$\Delta S = 0.14 S \Delta t + 0.20 S \varepsilon \sqrt{\Delta t}$$

Monte Carlo simulation

- Suppose
  - the current stock price is 20
  - the expected return from a stock is 14% (per year)
  - the volatility in returns is 20% (per year)
  - time steps (Δt) are in 0.01 years

- Simulation 1: If ε =1.0 (for 1 time step), then

$$\Delta S_1 = 0.14 \times 20 \times 0.01 + 0.20 \times 20 \times 1.0 \times \sqrt{0.01} = 0.40$$

- Simulation 2:If ε =-1.0 (for 1 time step), then

$$\Delta S_2 = 0.14 \times 20 \times 0.01 + 0.20 \times 20 \times (-1.0) \times \sqrt{0.01} = -0.40$$

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Monte Carlo simulation

- The current spot price is 20
- For price simulation number 1 the spot price in 1 time step is 20 + 0.40 = 20.40
- For price simulation number 2 the spot price in 1 time step is 20 - 0.40 = 19.60

Characteristics of energy/commodity prices

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Characteristics of energy prices

- High volatility (crude oil, natural gas, power)
- Jumps/spikes, regime switching
- Mean reversion
- Seasonality (price, volatility)
- Samuelson effect
- Stochastic volatility, volatility smile, volatility surface

- Distribution (leptokurtosis, skewness)

- Why is knowledge about price behaviour important?

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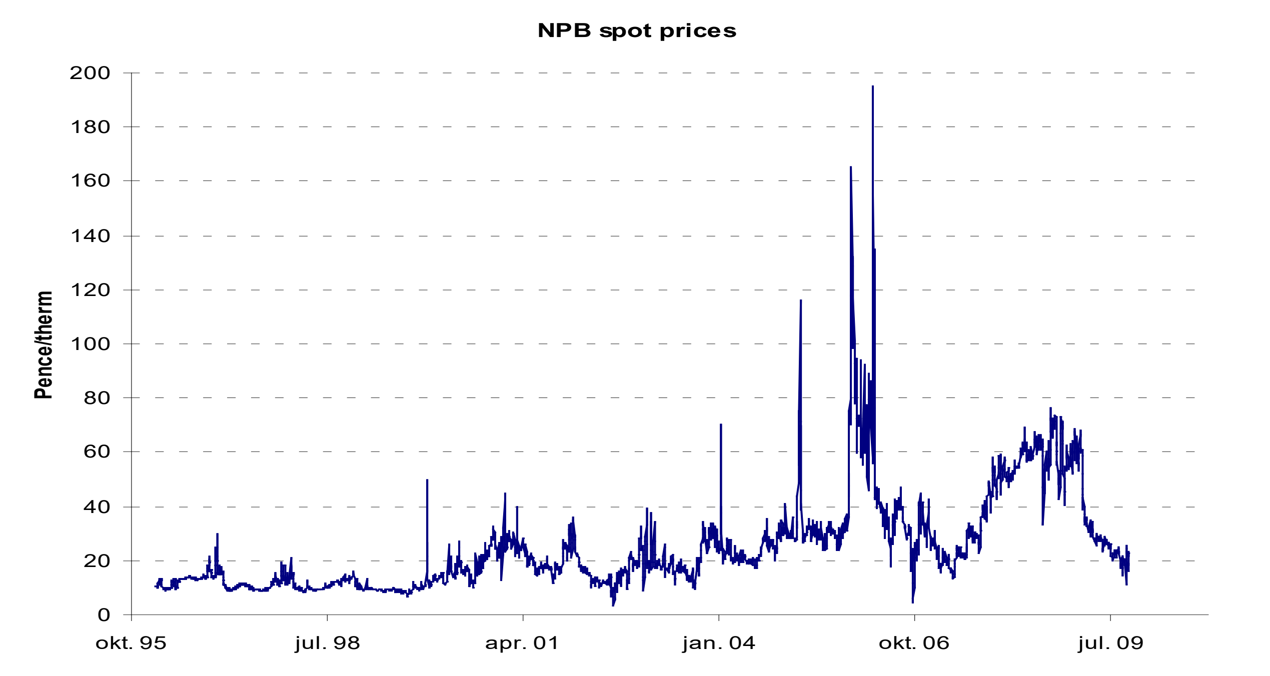
High volatility



Volatility is a measure of price fluctuations

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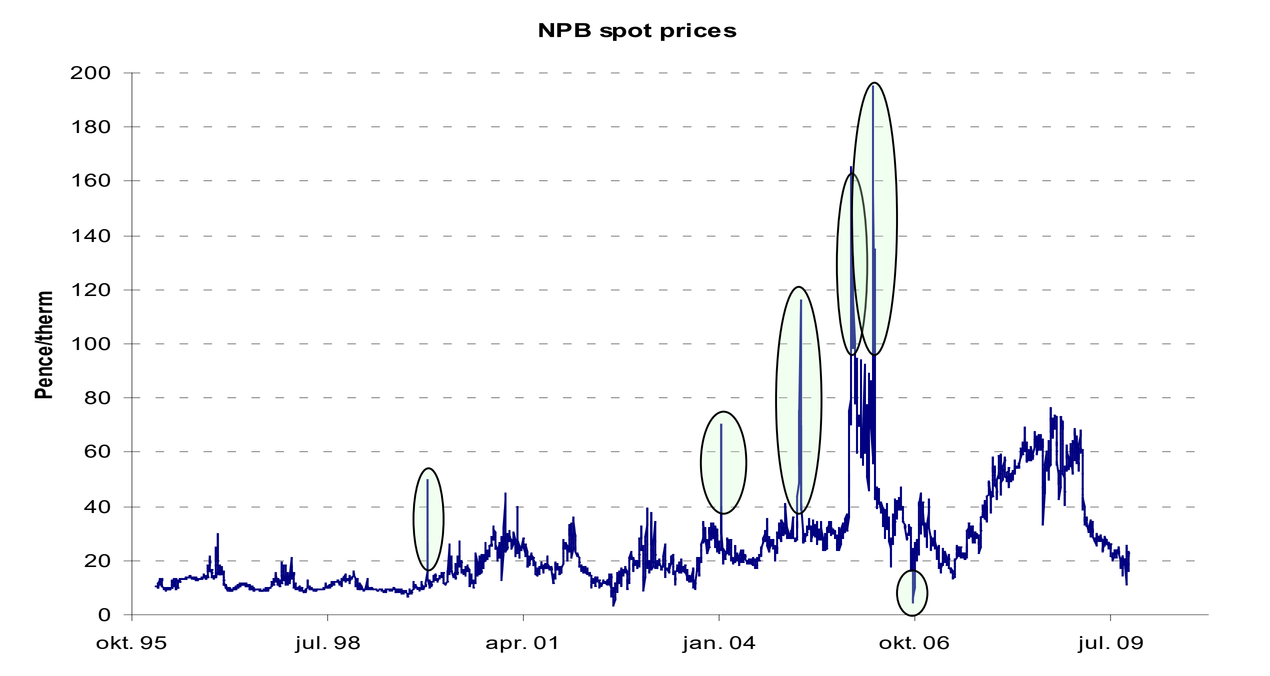
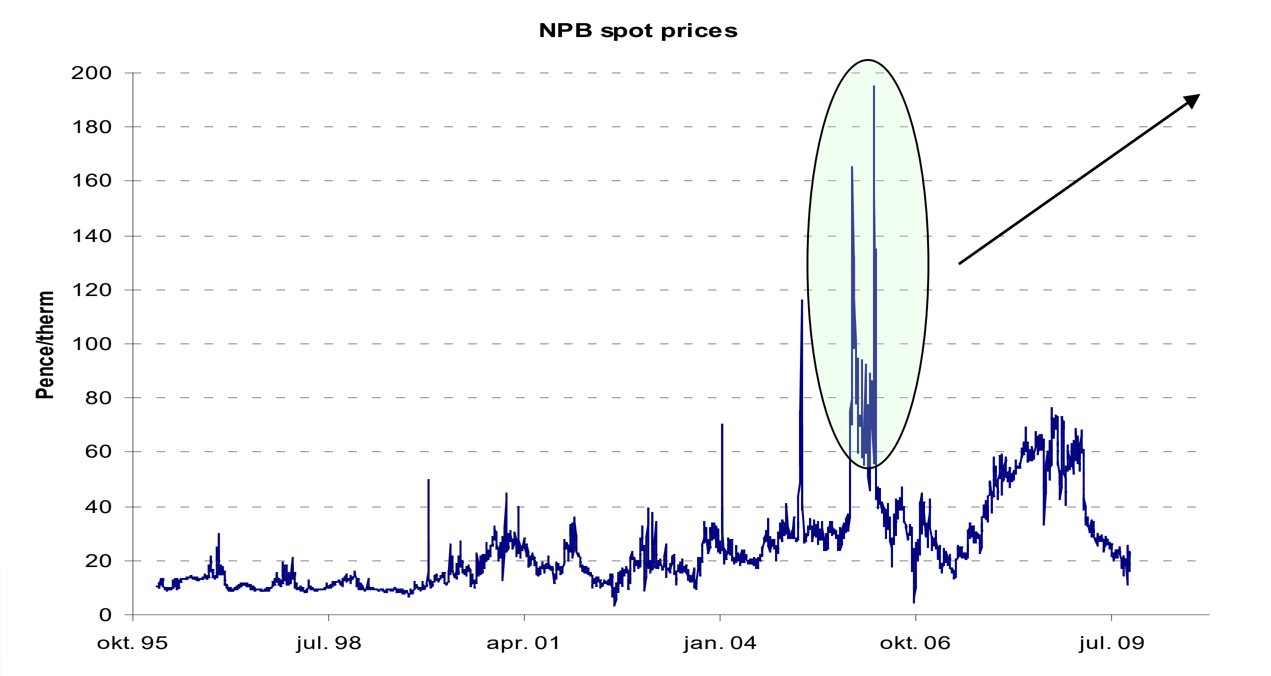
Jumps / price spikes



Often measured as price change > 3σ

Regime switching

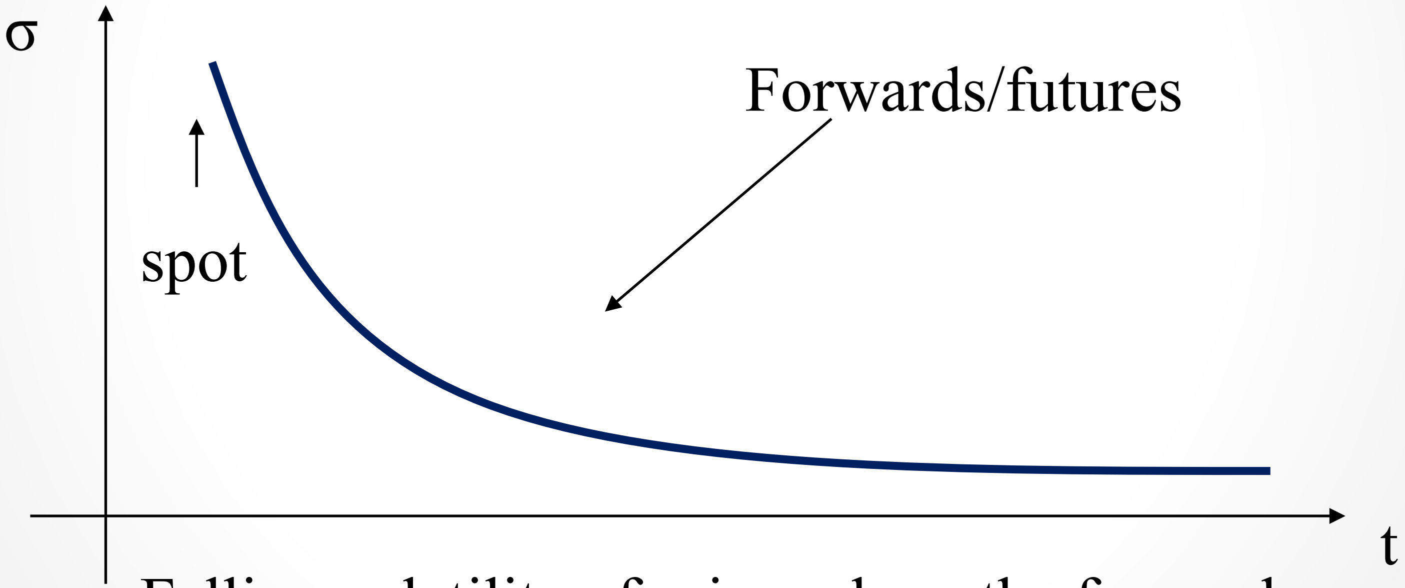


Regime switch?

Structural change of volatility, e.g. going from a regime of low volatility to a period of high volatility

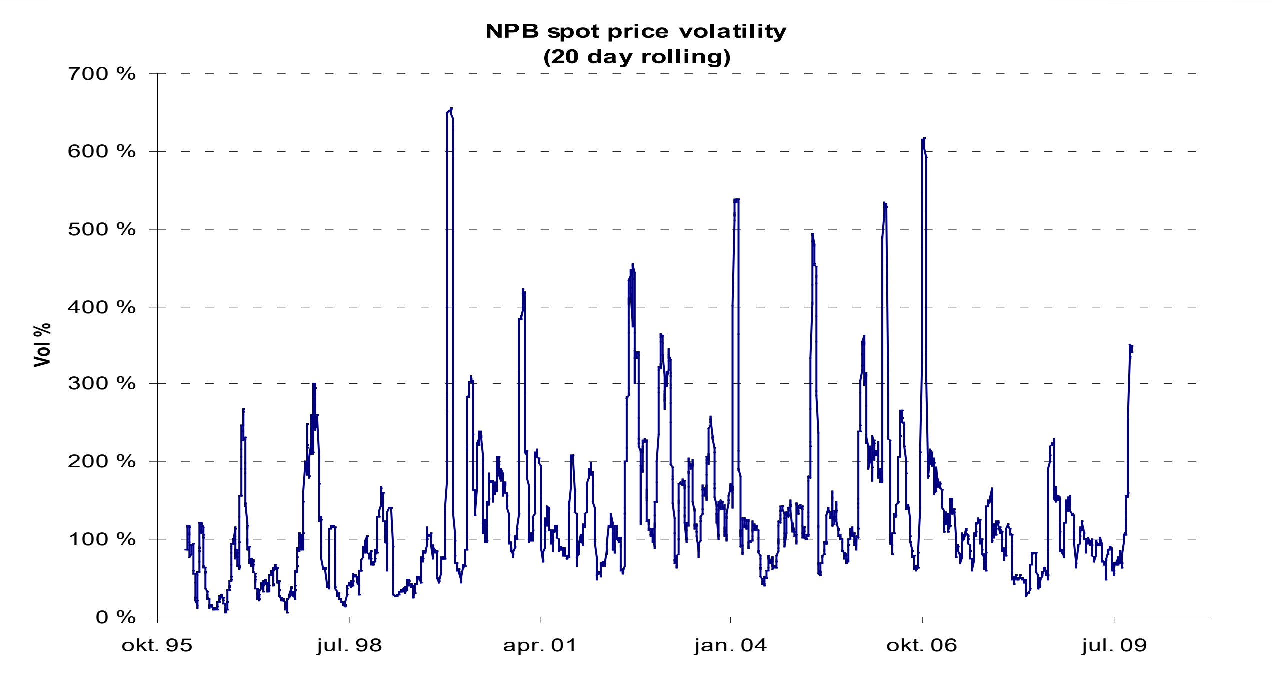
Samuelson effect

- Falling term structure of volatility



Falling volatility of prices along the forward curve

Stochastic volatility



Volatility changes over time in an uncertain way

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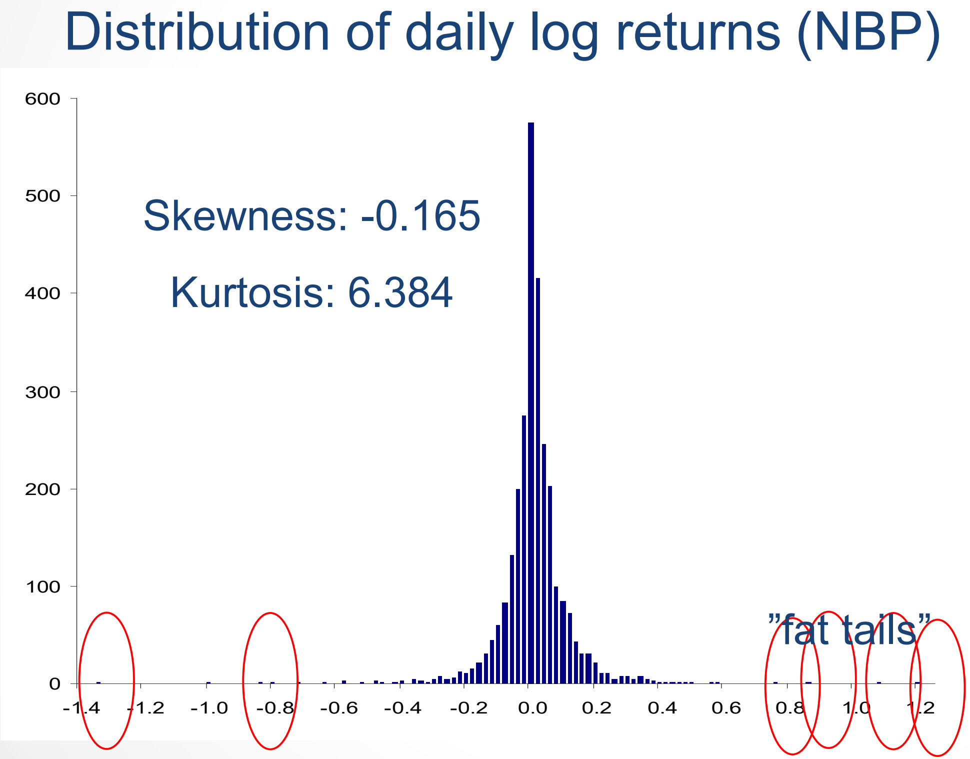
Distribution of log returns



- Normal distribution ?
    - Skewness (negative), kurtosis (leptokurtic)
- Including fat tails in the calibration will overestimate the standard deviation (volatility)
- Excluding fat tails will underestimate the true variability in prices
- Volatility including fat tails: 176%
- Volatility excluding fat tails: 149%

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Modeling commodity prices

Price modeling

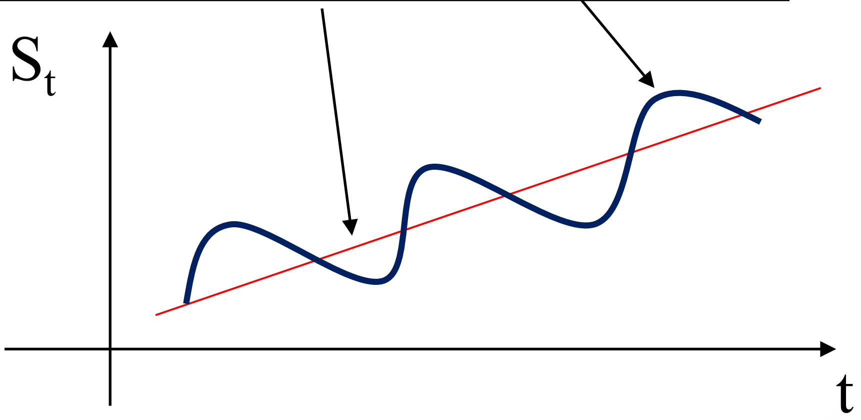
- Issues

1. Price forecasting vs risk modeling
2. Fundamental model vs Reduced form
3. Risk neutral prices?
4. Spot or spot-forward models
5. Simple models or advanced models
6. What is the underlying asset?
7. Calibration of parameters

Spot price model: GBM

- Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$



- Downside: does not capture important characteristics of energy prices (especially mean reversion)

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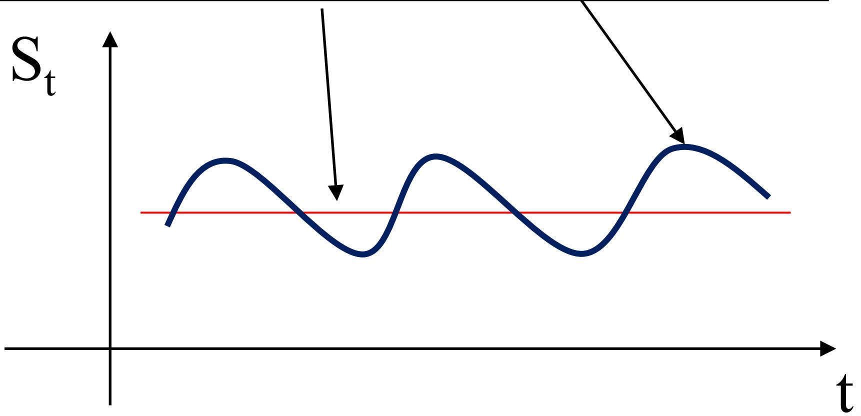
Spot price model: mean reversion

- Mean reversion

$$\boxed{dS_t = drift + stochastic}$$



Adjustment of drift term

Spot price model: Schwartz

- Mean reversion model
  - Vasicek: Ornstein-Uhlenbeck

- The most used price process is the Schwartz model (1997)

$$\boxed{\frac{dS_t}{S_t} = k(\theta - \ln S_t)dt + \sigma dW_t}$$

mean reversion rate	Long-term mean

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Spot price model: Schwartz

- Schwartz model (1997)

$$\boxed{\frac{dS_t}{S_t} = k(\theta - \ln S_t)dt + \sigma dW_t}$$

- Let x = ln(S). Converting to returns (changes) (and using Ito's Lemma) the model becomes:

$$\boxed{dx_t = k\left(\theta - \frac{\sigma^2}{2k}\right)e^{kt}dt + \sigma e^{kt}dW_t}$$

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Modeling of other characteristics

- Jumps (jump-diffusion)
- Regime switching (jump-diffusion with regime switch)
- Stochastic volatility
- Seasonality

Jumps & seasons

- Jumps

$$\boxed{\frac{dS_t}{S_t} = k(\theta - \ln S(t))dt + \sigma dW_t + \kappa_t dQ_t}$$

Price jump parameter

- Seasonality

$$dX_t = (\alpha - \beta X_t)dt + \sigma dW_t$$

$$\boxed{\ln S_t = f(t) + X_t}$$

f(t): sinus, cosinus

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Forward curve modeling

- The most common approach to forward curve modeling is:

$$\boxed{\frac{dF(t,T)}{F(t,T)} = \sum_j \sigma_j(t,T)\,dW_t^j}$$

volatilities are a
function of time

- Modelling forward curves can be done using Principle Component Analysis (PCA)

Numerical methods

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The Bigger picture

$$\frac{dS}{S} = \mu dt + \sigma dW$$

$\downarrow$ discretization

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

Monte Carlo simulation
Using this equation to create random price paths

Volatility surface
Volatility is often assumed to be constant. This chapter describes how this is not the case

Volatility estimation
This chapter describes methods for calculating the volatility of the underlying asset

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Numerical methods

- Numerical procedures are used when analytic results such as the Black-Scholes-Merton formulas do not exist

- Alternative 1: Trees
  - Binomial trees
  - Trinomial trees

- Alternative 2: Monte Carlo simulation
  - Stock price process
  - Energy price process (simple)
  - Calibration of parameters

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Numerical methods

- Black-Scholes is an exact formula (analytic solution)
- When exact formulas are not available, numerical procedures can be used
- Monte Carlo simulation are used for valuing derivatives where the payoff is dependent on the history of the underlying variable (asian option), or where there are several underlying variables (spread options)
- Trees are usually used for American options and other derivatives where the holder has early exercise rights prior to maturity

Binomial trees

- Building binomial trees
- Working backwards through the tree
- Expressing the approach algebraically
- Estimating Delta and other greeks
- Using the binomial tree for options on indices, currencies and futures contracts
  - With and without dividends

Control variate technique

- A technique called the control variate technique can improve the accuracy of the pricing of an American option
- Use the same tree to calculate both the value of the American option and (f_A) and the value of the corresponding European option (f_E)
- Also calculate the Black-Scholes value ($f_{B\text{-}S}$)
- The error given by the tree when valuing the European option is assued equal to that given by the tree when valuing the American option

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Control variate technique

- The estimate of the price of the American option is then

  $(f_A)+(f_{B\text{-}S})-(f_E)$

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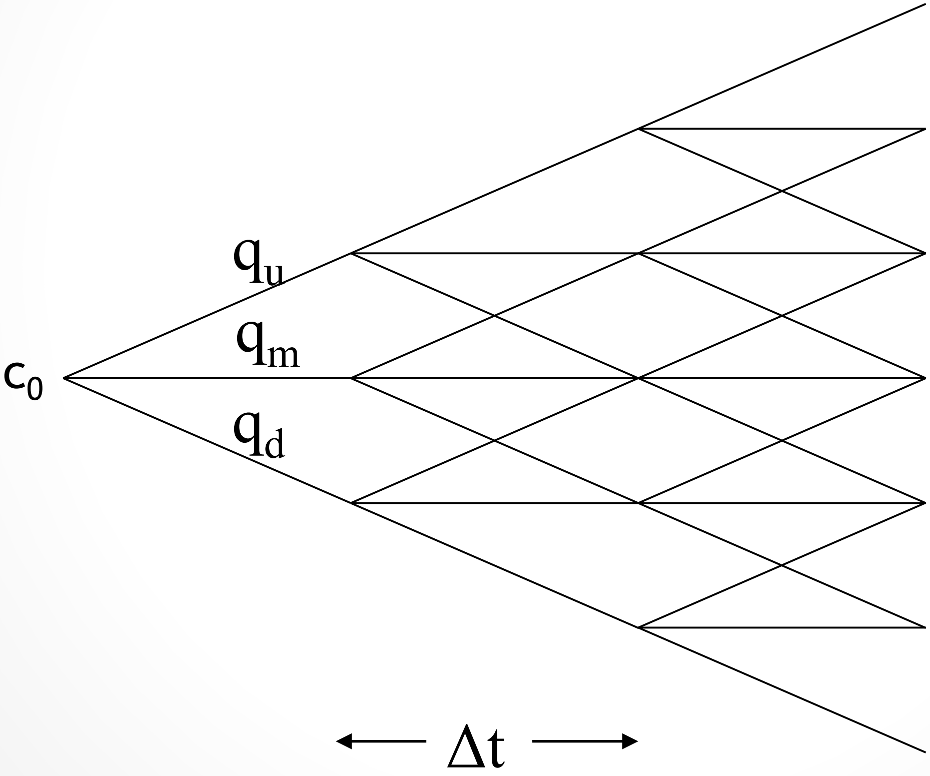
Trinomial trees: Alternative procedure for constructing trees



$$u = e^{\sigma\sqrt{3\Delta t}}$$

$$d = \frac{1}{u}$$

$$m = 1$$

$$q_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \delta - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

$$q_m = \frac{2}{3}$$

$$q_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \delta - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

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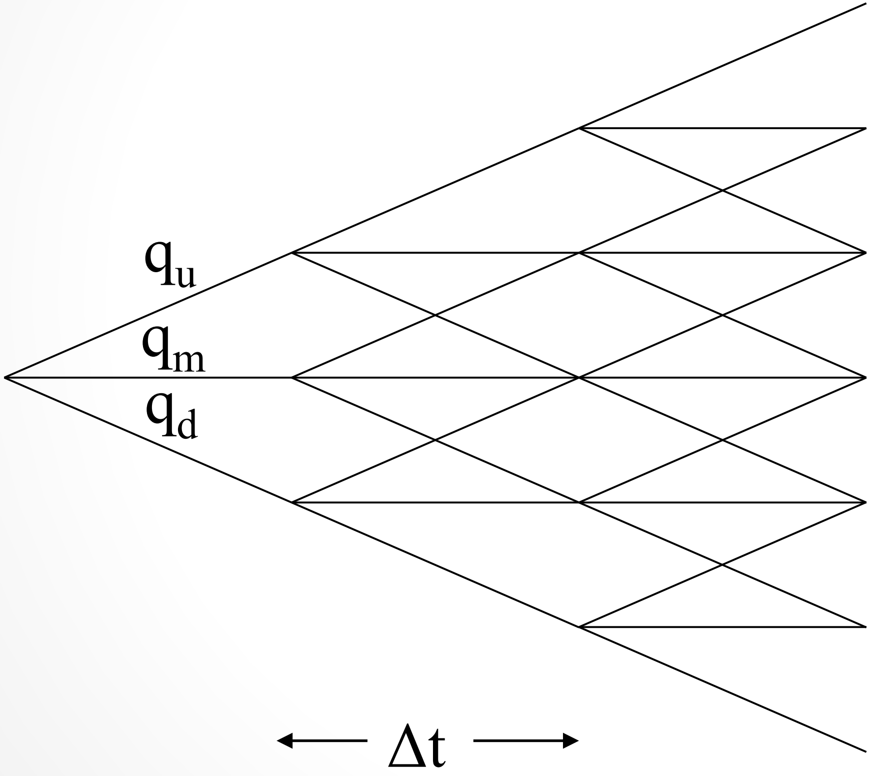
Trinomial trees



- Calculations analogous to binomial tree
- Work from end of the tree to the beginning
- At each node we calculate the value of exercising and value of continuing
- The value of continuing is

$$e^{-r\Delta t}(q_u c_u + q_m c_m + q_d c_d)$$

Monte Carlo simulation

- The Monte Carlo simulation method creates sample paths to obtain the expected payoff in a risk-neutral world and then discount this payoff at the risk-free rate
  1. Sample a random path for S in a risk-neutral world
  2. Calculate the payoff from the derivative
  3. Repeat 1. and 2. to get many sample values of the payoff from the derivative in a risk-neutral world
  4. Calculate the mean (average) of the sample payoffs to get an estimate of the expected payoff in a risk-neutral world
  5. Discount the expected payoff at the risk-free rate to get an estimate of the value of the derivative

Monte Carlo simulation

- First of all we need a price process we can use. In order to do that we need to complete 2 steps first:

- 1. We need to convert a ‘risky’ continuous time price into a risk neutral price process
    - This enables discounting using the risk free rate

- 2. We need to ‘discretize’ the price process, i.e. convert the continuous time model into a discrete time model. Monte Carlo simulation is carried out in discrete time

1. Risk neutral price process

- The process followed by a stock in a risky world is:

$$dS = \mu S dt + \sigma S dW \qquad \text{(GBM)}$$

- The process followed by a stock in a risk-neutral world is

$$dS = rSdt + \sigma S d\hat{W}$$

2. Discretization

- Continuous time

$$dS = rSdt + \sigma S d\hat{W}$$

- Discrete time
- To simulate the path followed by S, we divide the life of the derivative into N short intervals of length *dt*

$$\underbrace{S(t+\Delta t) - S(t)}_{\Delta S} = rS(t)\Delta t + \sigma S(t)\varepsilon\sqrt{\Delta t}$$

Logreturns

- In practice, it is more accurate to simulate ln S, rather than S

$$\ln S(t+\Delta t) - \ln S(t) = \ln\left(\frac{S(t+\Delta t)}{S(t)}\right)$$

This is the logarithmic return on S

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Price process for logreturns

Risk neutral stochastic process for price ***level***

$$dS = rSdt + \sigma Sd\hat{W}$$

Risk neutral stochastic process for price ***return***

$$d\ln S = (r - \frac{1}{2}\sigma^2)dt + \sigma d\hat{W}$$

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Discrete stochastic process for logreturns

- Continuous time

$$d \ln S = (r - \frac{1}{2}\sigma^2)dt + \sigma d\hat{W}$$

- Discrete time
- To simulate the path followed by S, we divide the life of the derivative into N short intervals of length Δt

$$\ln S(t+\Delta t) - \ln S(t) = (r - \frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

$$S(t+\Delta t) = S(t)e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}}$$

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Monte Carlo simulation

Day 0 S_0

Day 1 $S(t+\Delta t) = S_1 = S_0 e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon(1)\sqrt{\Delta t}}$

Day 2 $S_2 = S_1 e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon(2)\sqrt{\Delta t}}$

Day 3 $S_3 = S_2 e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon(3)\sqrt{\Delta t}}$

. .

. .

Day T $S_T = S_{T-1} e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon(T)\sqrt{\Delta t}}$

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Monte Carlo simulation

Day 0 S_0

Day 1 $S(t+\Delta t) = S_1 = S_0 e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon(1)\sqrt{\Delta t}}$

Day 2 $S_2 = S_1 e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon(2)\sqrt{\Delta t}}$

Day 3 $S_3 = S_2 e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon(3)\sqrt{\Delta t}}$

. .

. .

Day T $S_T = S_{T-1} e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon(T)\sqrt{\Delta t}}$

Draw new random number for each day

This creates 1 price path. Repeat process to create many price paths

Obtaining ε’s:Random draws

To create the stochastic process we need to draw 1) random numbers and 2) draw from a specific distribution

1. Random draws
   In Excel: RAND(): draws random numbers between 0 and 1

2. We need log returns to be normally distributed between $-\infty$ and $+\infty$, and with mean 0 and standard deviation 1 (i.e. a standard normal distributed variable)
   In Excel: NORM.S.INV() [Excel: NORMSINV]

Combined, we get NORM.S.INV(RAND()), and this can be used to find ε

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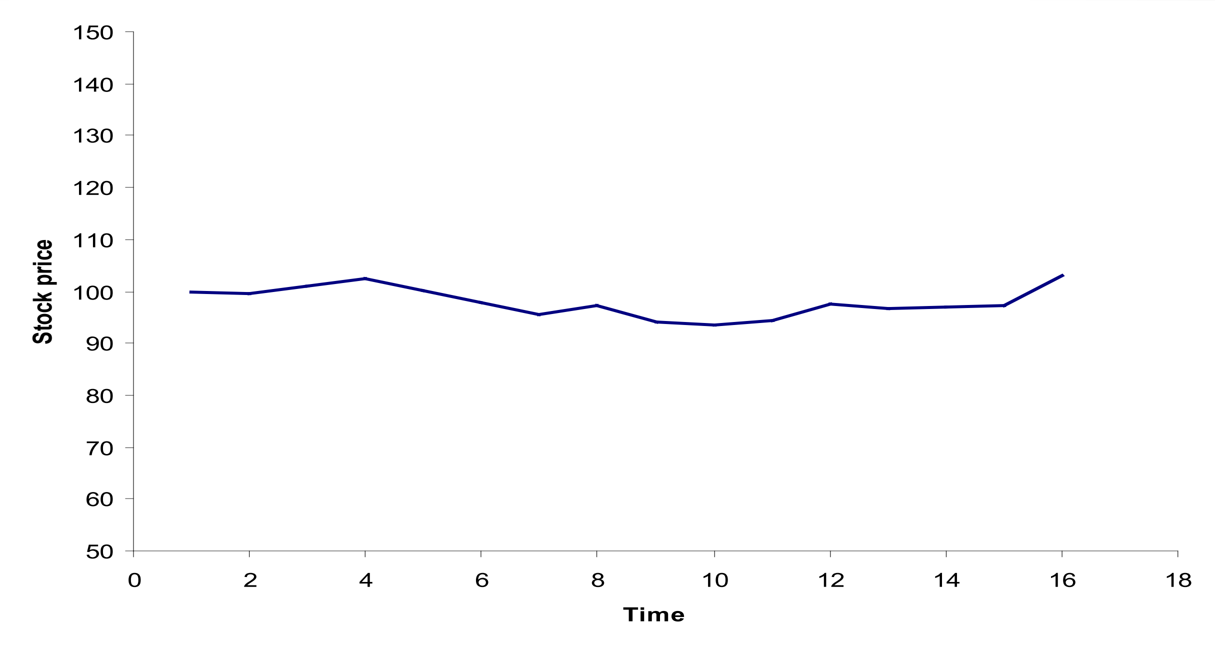
Monte Carlo simulation 1 path



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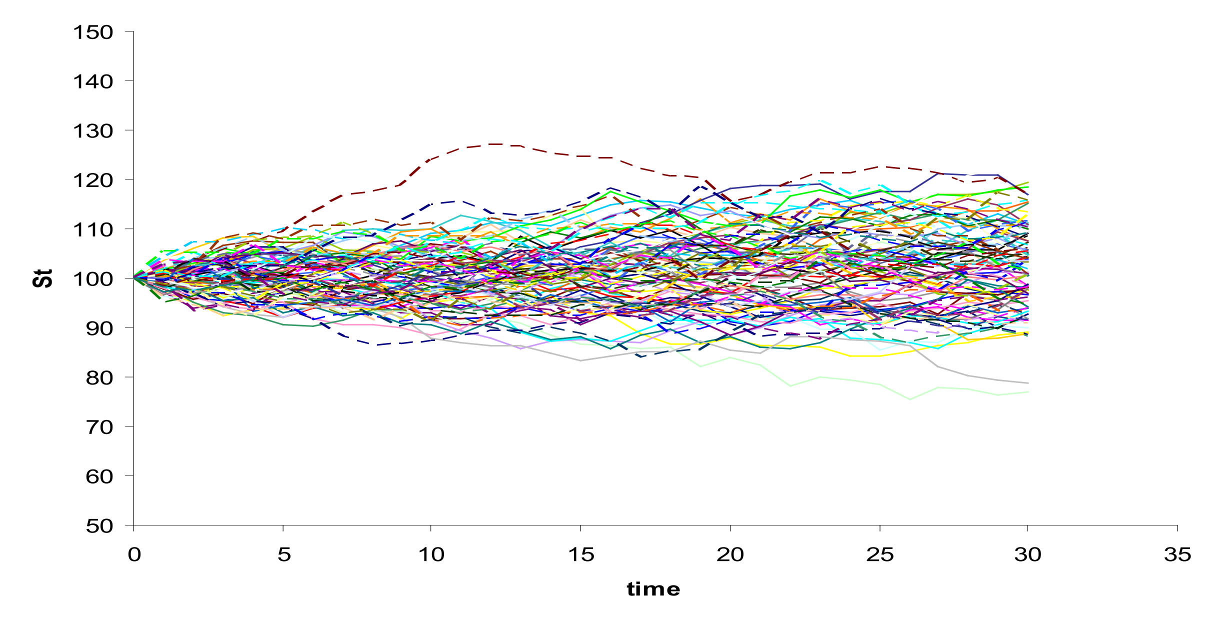
Monte Carlo simulation 100 paths



Monte Carlo simulation

- The Monte Carlo simulation method creates sample paths to obtain the expected payoff in a risk-neutral world and then discount this payoff at the risk-free rate
  1. Sample a random path for S in a risk-neutral world
  2. Calculate the payoff from the derivative
  3. Repeat 1. and 2. to get many sample values of the payoff from the derivative in a risk-neutral world
  4. Calculate the mean (average) of the sample payoffs to get an estimate of the expected payoff in a risk-neutral world
  5. Discount the expected payoff at the risk-free rate to get an estimate of the value of the derivative

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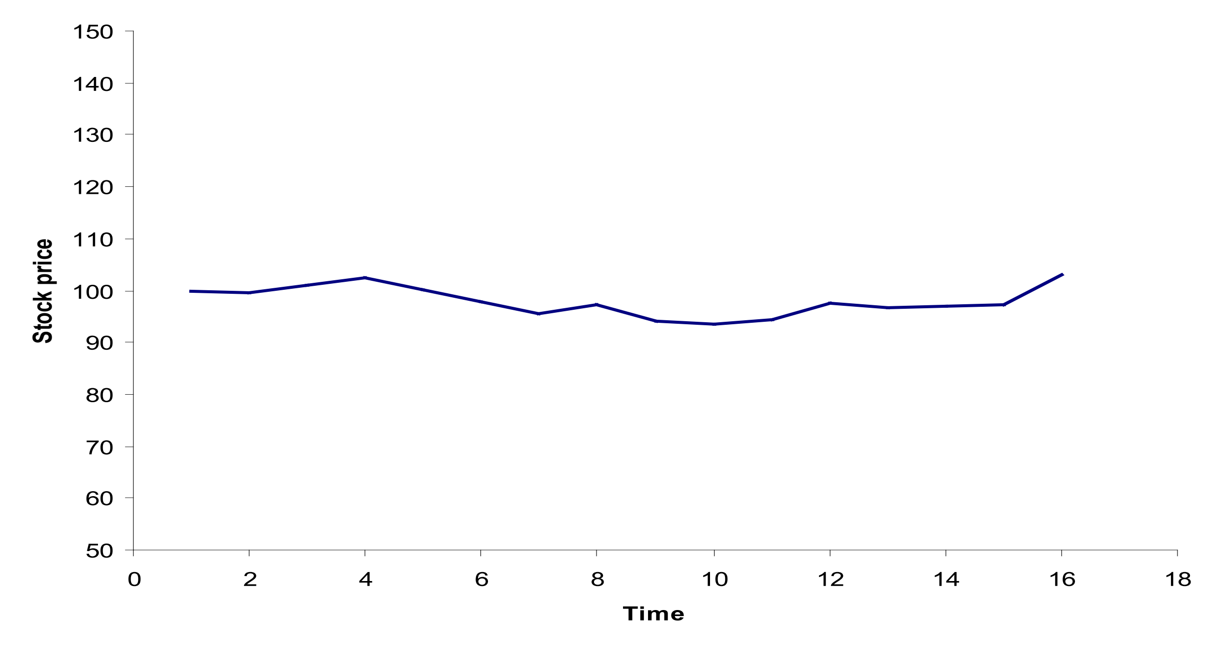
Step 1. Sample price paths



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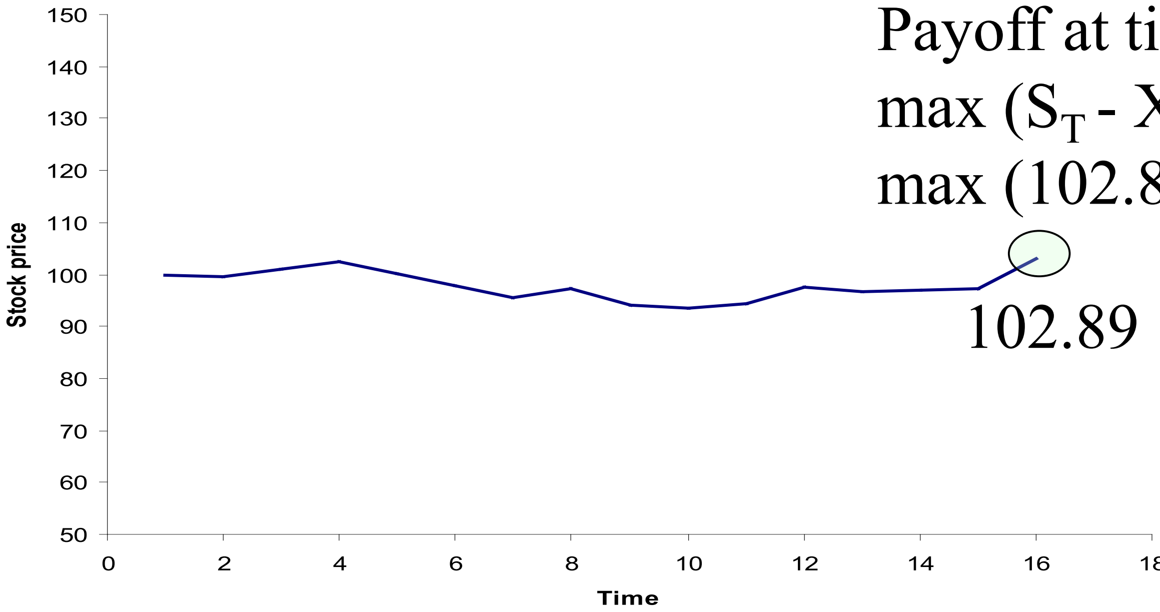
Step 2. Calculate payoff from derivative



Payoff at time T:

max (S_T - X,0)

max (102.89-X,0)

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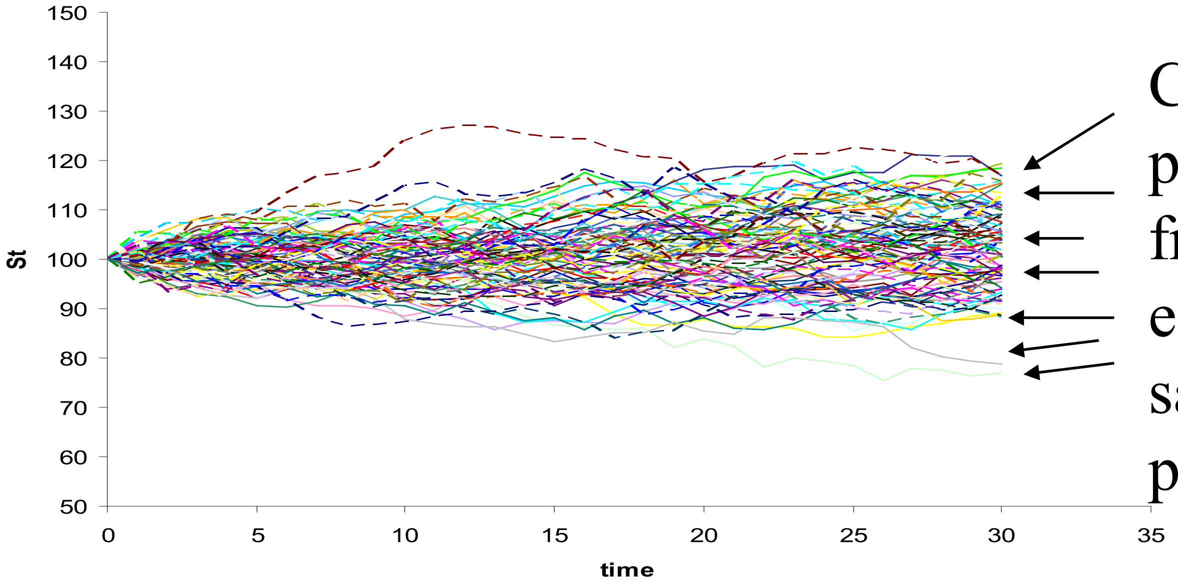
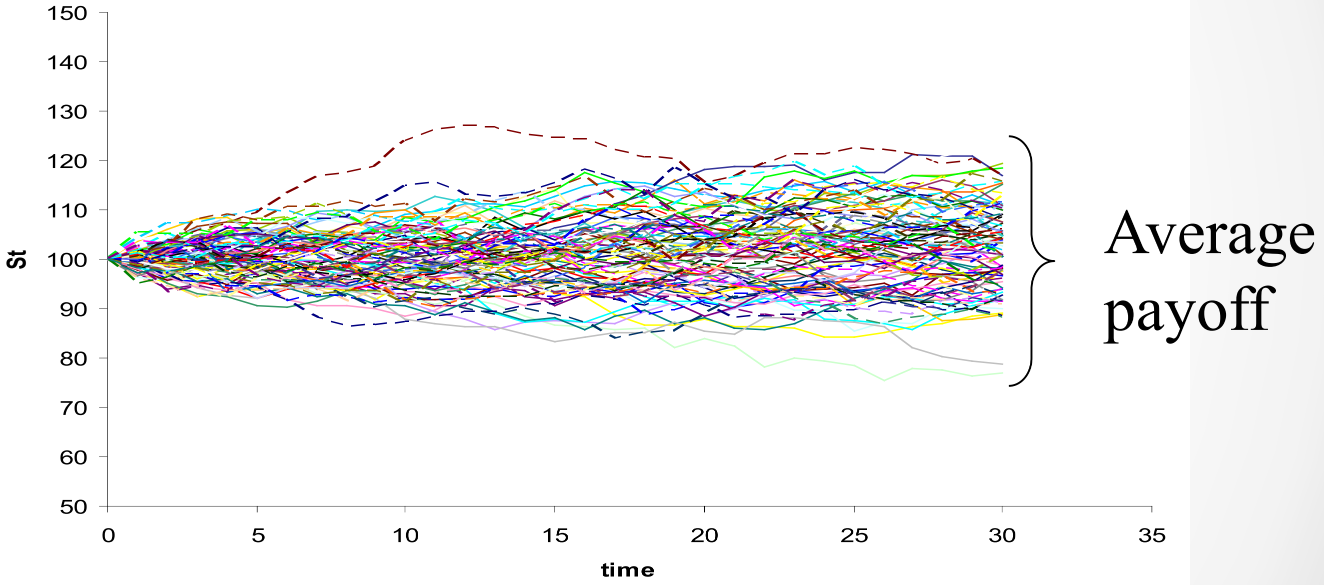
Step 3. Repeat steps 1. and 2.



Calculate payoffs from each sample path

Step 4. Calculate the average payoff



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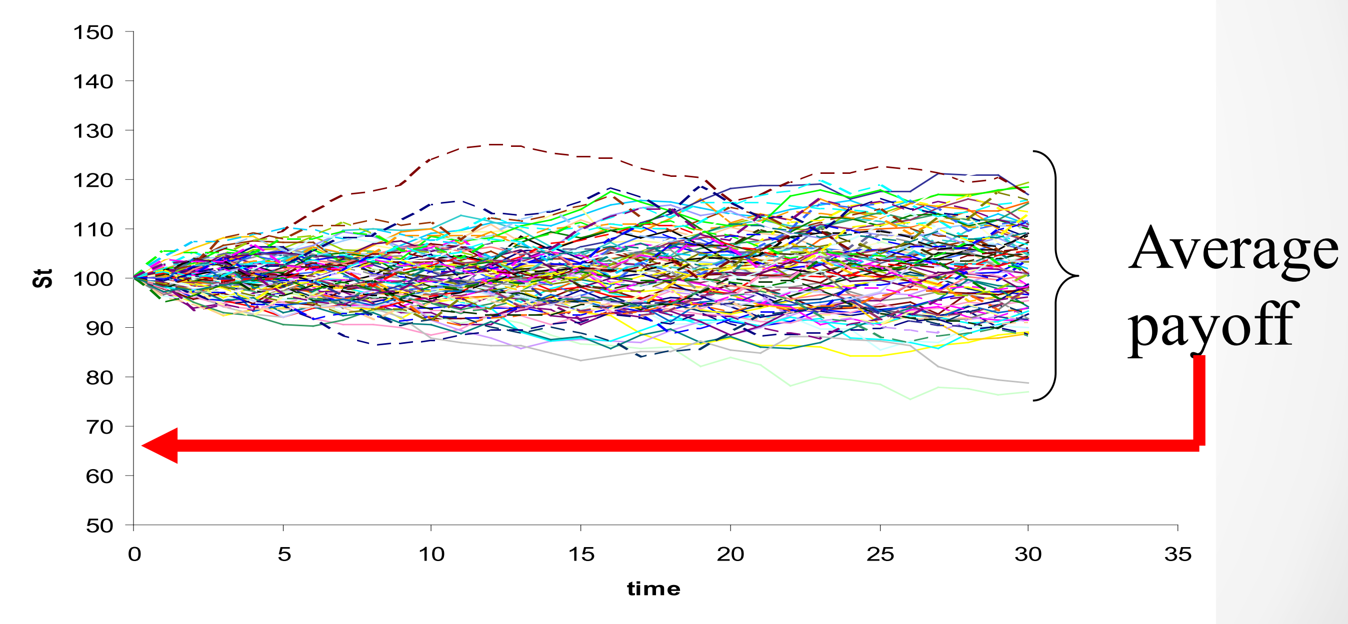
Step 5. Discount average payoff to today



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Simulating mean-reverting prices

- Mean reversion model
  - Vasicek: Ornstein-Uhlenbeck

- The most used price process is the Schwartz model (1997)

$$\boxed{\frac{dS_t}{S_t} = k(\theta - \ln S_t)dt + \sigma dW_t}$$

mean reversion rate

Long-term mean

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Simulating correlated prices

- When simulating 2 or more correlated prices (Price 1 and Price 2), a correlation coefficient must be used
- The first step is to generate 2 random variables

ε_1 = N~(0,1)

ε_2 = N~(0,1)

$$\varepsilon_3 = \varepsilon_1 \rho + \varepsilon_2 \sqrt{1-\rho^2}$$

- Then for simulating Asset 1: use $\boldsymbol{\varepsilon_1}$
- And for simulating Asset 2: use $\boldsymbol{\varepsilon_3}$ (from ε_1 and ε_2)

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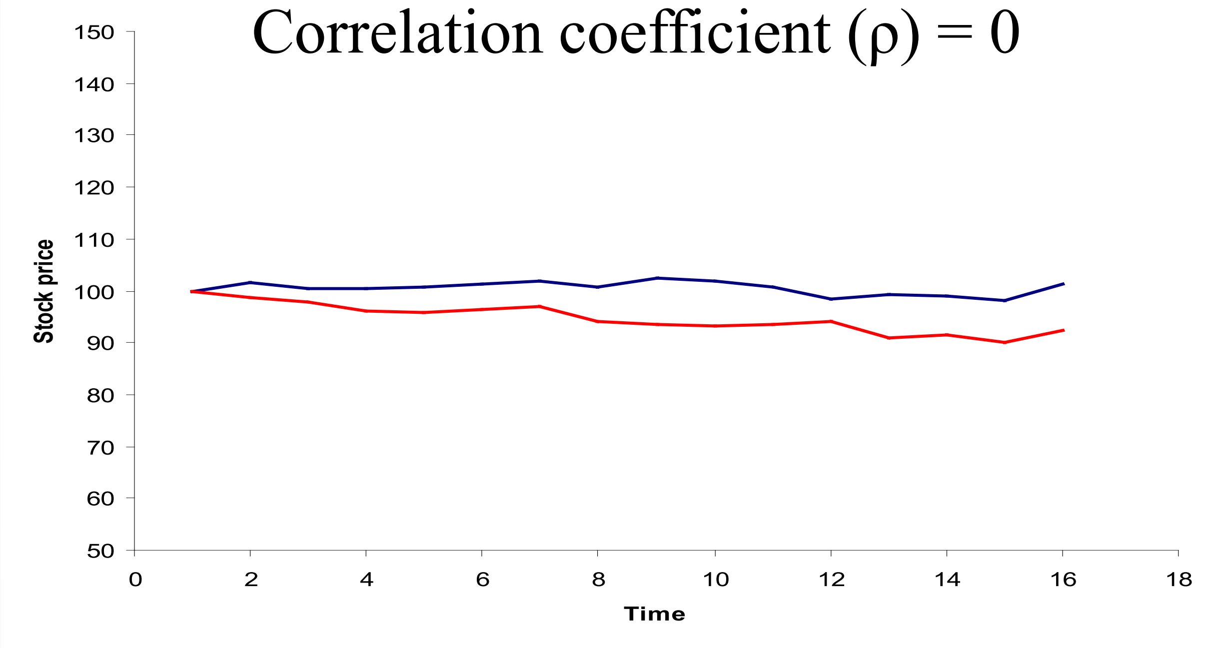
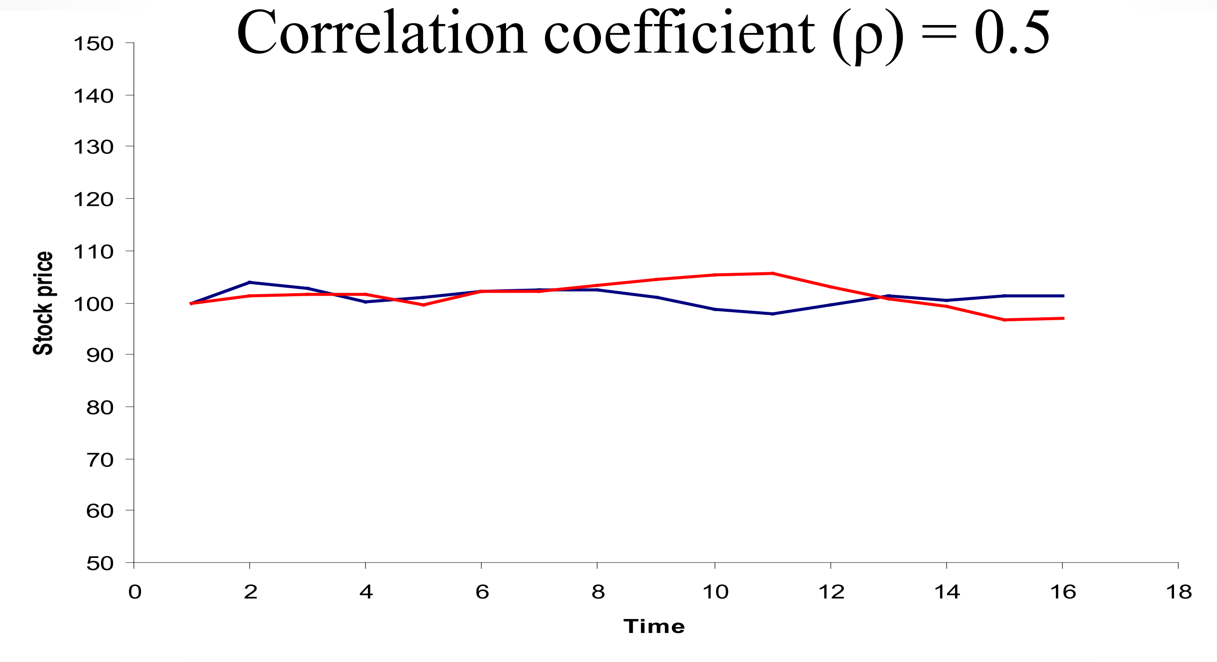
Simulating correlated prices



Simulating correlated prices



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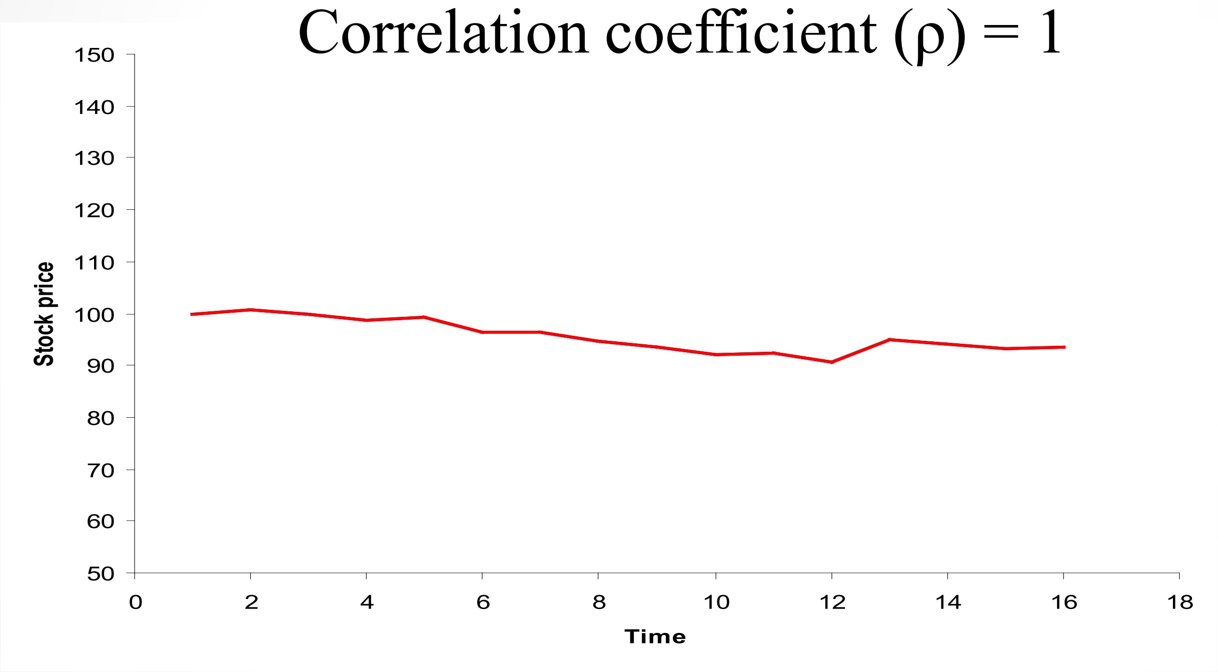
Simulating correlated prices



Which process?

- Check behaviour of underlying prices
- Choose appropriate mathematical price process
- Discretise price process
- Calibrate input parameters (value drivers)

Volatility calibration

$$dS = \mu S dt + \sigma S dW$$

Volatility & volatility estimation

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Calibration of parameters

- Volatility estimation (needed for all option pricing)
  - Simple weighted
  - EWMA
  - ARCH
  - GARCH
- Correlation (needed to price spread options)
- Mean reversion rate (needed to price options on energy assets)

Volatility

- Volatility is not observable (not like e.g. stock prices)
- Volatility has to be estimated
- Numerous methods for estimating volatility
  - Implied volatility
  - Forecasts based on historical volatility
    - Simple approach
    - Rolling window
    - Exponential moving average (EWMA)
    - Generalised Autorregressive Conditional Heteroscedasticty (GARCH)

Estimating volatility using option prices: Implied volatility

- Implied volatility: Volatilities implied by option prices observed in the market
- Model-implied: estimated using an analytic formula, e.g. Black-Scholes
- Model-free implied: estimated from option prices without using a specific pricing model (e.g. The VIX index (aka 'The fear index'))

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Estimating volatility using historical prices: Simple historical estimation

- 1. Get historical prices (close prices, last): P1, P2, ...Pn
- 2. The continuously compounded return (log returns) from yesterday (day *i-1*) to today (day *i*) is

  $u_i = \ln((S_i)/S_{i-1})$

- 3. Calculate the daily variance rate of log returns using the most recent observations of u_i

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \overline{u})^2$$

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Estimating volatility using historical prices: Simple historical estimation

- 4. Often a simplified version is used

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2$$

$\bar{u}$ is assumed to be 0
m-1 replaced by m

- 5. Annualise to get yearly volatility

$$\sigma_{annual}^2 = \sigma_{daily}^2 h$$

- h are the number of time periods per year
- If daily observations are used, h=252
- If weekly observations are used, h=52

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Volatility: Rolling window

- Often rolling windows are used
- A 20-day rolling volatility is calculated using the formula on the previous slide, using a 20 day estimation window
  - 20 working days ~ 1 month
- Other rolling windows can be used
  - Short window: volatility changes quickly and more eratic
  - Long-window: volatility changes slowly

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Weighting schemes

- The simple approach weights each observation equally
- However, since we want to estimate current volatility, it makes sense to give recent observations more weight
- We should therefore use a weighting scheme
- A model that does this is

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

α_i's are weights, $\sum_{i=1}^{m} \alpha_i = 1$

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Exponentially Weighted Moving Average (EWMA)

- If we let the weights decrease exponentially as we move back in time, we get the EWMA model

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-i}^2$$

- Example, let λ=90% $\quad \sigma_n^2 = 0.90 \times \sigma_{n-1}^2 + (1-0.90)u_{n-i}^2$

- If yesterday's crude oil volatility was 1% (daily volatility), the return on crude oil prices was 2%, what is today's volatility?

$$\sigma_n^2 = 0.90 \times (0.01)^2 + (1-0.90)(0.02)^2 = 0.00013$$

$$\sigma_n = \sqrt{0.00013} = 1.14\%$$

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Autoregressive Conditional Heteroscedasticity (ARCH) model

If we include a long run average volatility, V_L, we arrive at the ARCH model

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}{}^2 \qquad \gamma + \sum_{i=1}^{m} \alpha_i = 1$$

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Generalised AutoRegressive Conditional Heteroscedasticity (GARCH)

- The equation for GARCH (q,p) is

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{q} \alpha_i u_{n-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{n-i}^2$$

- Using 1 lag, GARCH (1,1) is

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \qquad \gamma + \alpha + \beta = 1$$

Today's volatility

Long term volatility

Yesterday's return

Yesterday's volatility

GARCH and EWMA

- If γ=0, α=1-λ, and β=λ, the GARCH (1,1) model is reduced to EWMA

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

$$\downarrow$$

$$\sigma_n^2 = (1-\lambda) u_{n-1}^2 + \lambda \sigma_{n-1}^2$$

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Generalised AutoRegressive Conditional Heteroscedasticity (GARCH)

- By setting $\omega = \gamma V_L$

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

$$\downarrow$$

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

- This is the model that we will try to estimate

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Expanding GARCH (1,1)

(1) $\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$ today

(2) $\sigma_{n-1}^2 = \omega + \alpha u_{n-2}^2 + \beta \sigma_{n-2}^2$ yesterday

Putting (2) into (1) gives

$$\sigma_n^2 = \omega + \beta\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \beta^2 \sigma_{n-2}^2$$

Repeating with σ^2_{n-2} we get

$$\sigma_n^2 = \omega + \beta\omega + \beta^2\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \alpha\beta^2 u_{n-3}^2 + \beta^3 \sigma_{n-3}^2$$

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Expanding GARCH (1,1)

Repeating with σ^2_{n-2} we get

$$\sigma_n^2 = \omega + \beta\omega + \beta^2\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \alpha\beta^2 u_{n-3}^2 + \beta^3\sigma_{n-3}^2$$

β can be interpreted as a 'decay rate' since we require that β <1

$$\underset{\text{today's volatility}}{\sigma_n^2} = \underbrace{\omega + \beta\omega + \beta^2\omega}_{\text{Long-run volatility}} + \underbrace{\alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \alpha\beta^2 u_{n-3}^2}_{\text{'shocks'}} + \underset{\text{recent volatility}}{\beta^3\sigma_{n-3}^2}$$

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Estimating GARCH

- 1. In Excel
- 2. Statistical software
  - R, Stata, Matlab, Python, etc..

GARCH: Hull approach

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

In order to calculate today's volatility using a GARCH (1,1) model, we need to estimate the parameters: ω,α, and β.

We use the Maximum Likelihood approach to estimate the parameters

Basic principle: The aim of maximum likelihood estimation is to find the parameter values that makes the observed data most likely

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Maximum Likelihood: Basic principle

- Suppose we have a random sample of size 2 (y1 and y2) from a N(μ,1) distribution
- We know that the sample is drawn from a normal distribution with standard deviation 1, but we do not know the mean (μ)
- Our task is to estimate the mean
- As soon as we choose a value for μ, we know the complete probability density of the data, and we can calculate the probability of observing our sample data given that choice
- This is the sample likelihood

Maximum Likelihood: Basic principle

- Suppose we plug in a guess for μ, and calculate the sample likelihood (using a probability density function)
- Suppose it turns out that the value for the sample likelihood is very low
- We can conclude that our initial guess was wrong, inconsistent with the data, and we repeat with a another guess for μ
- We repeat until we arrive at the highest value for the sample likelihood (i.e. maximise the likelihood function)

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GARCH Maximum likelihood

- In terms of GARCH (1,1), we want to find the estimates for α, β, and ω which gives us the highest value for the likelihood function
- It can be shown that the likelihood function for GARCH (1,1) is

$$\text{Log likelihood function} = \sum_{i=1}^{m}\left[-\ln(v_i) - \frac{{u_i}^2}{2v_i}\right] \qquad v = \text{variance}$$

- The principle is that we calculate the log likelihood function repeatedly using different values for α, β, and ω
- We are looking for the combination of parameters that gives us the highest value for the log likelihood function

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GARCH estimation in R

- Many packages
- One useful packages is fGarch

- 1. Estimate GARCH (1,1)
  - garch1=garchFit(~garch(1,1),data="logreturn data", trace=F, include.mean=F)

- 2. Pull out parameters
  - summary(garch1)

- 3. Pull out σ_n^2
  - cat(garch1@sigma.t,sep="\n")

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Correlation

- The correlation coefficient between two variables X and Y can be defined as

$$E[(X-\mu_X)(Y-\mu_Y)]$$

- Let x and y be returns on X and Y

$$x_i = \frac{X_i - X_{i-1}}{X_{i-1}} \qquad y_i = \frac{Y_i - Y_{i-1}}{Y_{i-1}}$$

- The covariance between x and y is calculated as

$$\rho_{x,y} = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$$

- where

$$\operatorname{cov}(x,y) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}) \qquad \sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad \sigma_y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y})^2}$$

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Mean reversion estimation

$$\frac{dS_t}{S_t} = k(\theta - \ln S_t)dt + \sigma dW_t$$

We want to estimate this parameter (i.e. tells us how quickly prices return to the long run price level

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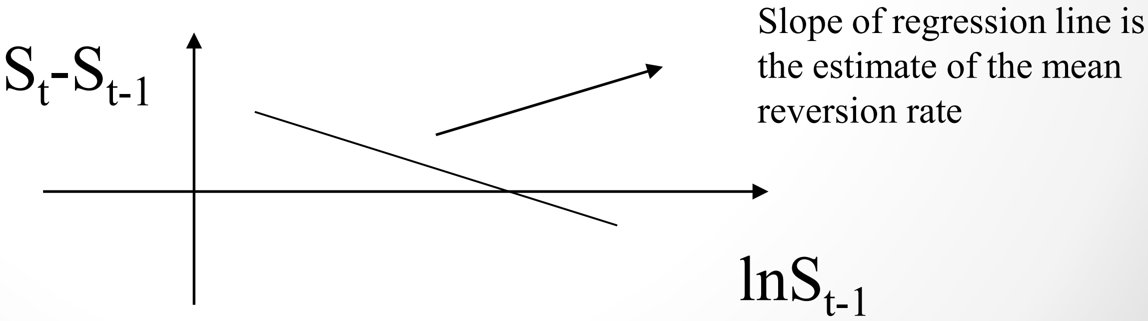
Calibration: Mean reversion rate

- An approach to estimating the mean reversion rate is to regress delta S on lagged lnS (t) ($\ln S_{t-1}$)
- If S(t) is high, then the change in price in the next time period should be negative
- If S(t) is low, then the change in price in the next time period should be positive



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Calibration: Mean reversion rate

- Estimate the following regression model

$$\Delta S_t = \alpha + \beta \ln S_{t-1} + \varepsilon \qquad \Delta S = S_t - S_{t-1}$$

- The coefficient on S_{t-1}, ß, will be the estimate of the mean reversion rate (daily)
- The coefficient will be negative (<0)
- It can be converted to annual mean reversion rate by multiplying with 250 days

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Relevant literature

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